$$\frac{dy}{dt} + g(t)\,y = b(t) \quad (*)$$

step 1: $\boxed{\mu(t) = e^{\int g(t)\,dt}}$ integrating factor.

step 2: $\boxed{\frac{d}{dt}(\mu(t)\,y)} = \mu(t)\cdot b(t) \Rightarrow \mu(t)\,y = \int \mu(t)\,b(t)\,d\mu$

$\parallel$

$\underline{\mu(t)\frac{dy}{dt} + \mu'(t)\,y}$

$$\Rightarrow \boxed{y = \mu(t)^{-1} \underbrace{\int \mu(t)\,b(t)\,dt}_{(C + k(t))}} = \frac{C}{\mu(t)} + \mu(t)^{-1}\cdot k(t)$$

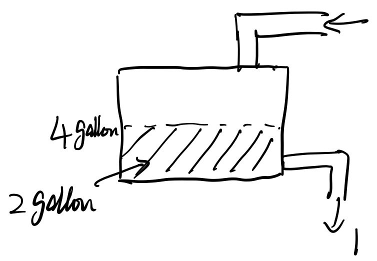
a particular solution to (*)

$C$ ↑ constant.

↑ gen. solution to the associated homogeneous eq.

$$\frac{dy}{dt} + g(t)\,y = 0.$$

Mixing Problem:



- 2 gallon per min
- concentration 0.5 pound/gallon

. initial salt in the tank = 2 pounds.

1 gallon per min

Q: What is the amount of salt when the tank is full?

volume at time $t$ $\parallel$ $2+t$

$y$ = amount of salt in the tank.

$$\boxed{\frac{dy}{dt} = \underbrace{\boxed{0.5\times 2}}_{\text{con. × Volume}} - \underbrace{\boxed{\frac{y}{2+t}\times 1}}_{\text{con. × vol}} = 1 - \frac{y}{2+t}, \quad y(0) = 2}$$

$\frac{d}{dt}$ salt $\parallel$ salt into the tank − salt out of the tank

$$\boxed{\frac{dy}{dt} + \frac{y}{2+t} = 1} \;.\; \mu(t) = e^{\int \frac{dt}{2+t}} = e^{\ln(2+t)} = (2+t).$$

$$\cdot\; \frac{d}{dt}\left((2+t)\,y\right) = 2+t \Rightarrow (2+t)\,y = \int (2+t)\,dt = 2t + \frac{t^2}{2} + C$$

$$\Rightarrow \quad y(t) = \frac{1}{2+t} \cdot \left( 2t + \frac{t^2}{2} + C \right).$$

$$y(0) = \frac{1}{2} \cdot (C) = 2 \Rightarrow C = 4$$

$$\Rightarrow y(t) = \frac{1}{2+t} \left( 2t + \frac{t^2}{2} + 4 \right)$$

The tank becomes full at $t=2$:

$$y(2) = \frac{1}{2+2} \left( 2 \times 2 + \frac{2^2}{2} + 4 \right) = \frac{1}{4} \cdot (4+2+4) = \frac{10}{4} = 2.5$$

---

Ex: $\dfrac{dy}{dt} = t + \dfrac{2y}{1+t}$

step0: $\underline{\dfrac{dy}{dt} - \dfrac{2}{1+t} \cdot y = t}$.

step1: $\displaystyle\int g(t)dt = \int -\frac{2}{1+t} dt = -2 \cdot \ln(1+t).$

$$\mu(t) = e^{\int g(t)dt} = e^{-2 \cdot \ln(1+t)} = \frac{1}{(1+t)^2}$$

step 2: $\dfrac{d}{dt} \left( \dfrac{1}{(1+t)^2} \cdot y \right) = \dfrac{t}{(1+t)^2}$

$$\Rightarrow \boxed{\frac{1}{(1+t)^2}} y = \int \boxed{\frac{t}{(1+t)^2}} dt = \ln|1+t| + \frac{1}{1+t} + \underline{C}$$

$$\int \frac{t+1-1}{(1+t)^2} = \int \frac{1}{1+t} - \int \frac{1}{(1+t)^2}$$

$$\Rightarrow y = (1+t)^2 \ln|1+t| + (1+t) + \underline{C \cdot (1+t)^2}$$

2nd order linear differential equation.

$$\boxed{\text{1st: } \frac{dy}{dt} + g(t)\, y = b(t).}$$

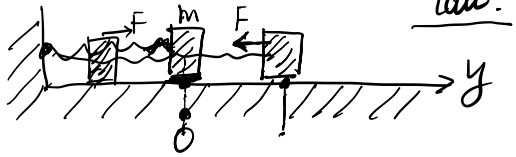
2nd: $$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\, y = \boxed{b(t)}$$

associated homogeneous linear diff. eq.:

$$\boxed{\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\, y = 0}$$

- Modeling the oscillation (mass-spring (- damper) system).

Hooke's law: $F = -k \cdot y$ $\qquad$ $k$ = Hooke's constant.



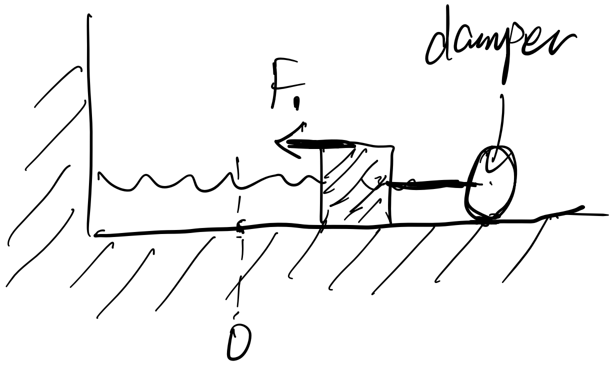
Newton's law: mass x acceleration = force.

$$\boxed{m \times \frac{d^2y}{dt^2} = F}$$

velocity: $v = \frac{dy}{dt}$, acceleration $= \frac{d}{dt}\frac{dy}{dt}$
$= \frac{d^2y}{dt^2}$

$$m \cdot \frac{d^2y}{dt^2} = -k \cdot y \Leftrightarrow \boxed{m \frac{d^2y}{dt^2} + k \cdot y = 0}$$

$$\frac{d^2y}{dt^2} + \frac{k}{m} y = 0.$$ homogeneous linear equation



$$F_1 = -ky$$

$$F_2 = -b \cdot \frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} = -k \cdot y - b \cdot \frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

---

Forced oscillation: extra force $f(t)$

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f(t).$$

$$\boxed{m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = \underbrace{f(t)}_{\uparrow}}$$

non-homogeneous.

$\boxed{\text{Linearity Principle}}$: general solution to a non-hom. linear diff. equation

$$y = \underbrace{y_h}_{\substack{\uparrow \\ \text{solution to associated} \\ \text{homogeneous lin. eq}}} + y_p \leftarrow \text{particular sol. to the non-hom. diff. equation.}$$

Ex: $\underline{\dfrac{d^2y}{dt^2} = 4y} \;.\; \Leftrightarrow \underline{\dfrac{d^2y}{dt^2} - 4y = 0.}$

$\underline{\boxed{\left(\dfrac{dy}{dt} = 4y \Leftrightarrow y = e^{4t}\right)}}$ $\quad \boxed{y(t) = e^{\lambda t}}$.

$$\frac{dy}{dt} = \lambda \cdot e^{\lambda t}, \quad \underline{\frac{d^2y}{dt^2} = \lambda^2 e^{\lambda t}}$$

$$\lambda^2 \cdot e^{\lambda t} - 4 \cdot e^{\lambda t} = 0 \overset{e^{\lambda t} \neq 0}{\Longleftrightarrow} \boxed{\lambda^2 - 4 = 0}$$

$$\Leftrightarrow \lambda = \pm 2.$$

$$\underline{\boxed{y_1(t) = e^{2t}, \; y_2(t) = e^{-2t}}}$$

$\Rightarrow y(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{-2t}$ general solution.

Ex: $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} - 7y = 0.$

$$y(t) = e^{\lambda t} \Rightarrow \quad \frac{dy}{dt} = \lambda \cdot e^{\lambda t}, \quad \frac{d^2y}{dt^2} = \lambda^2 \cdot e^{\lambda t}$$

$$\lambda^2 \cdot e^{\lambda t} - 6\lambda e^{\lambda t} - 7 \cdot e^{\lambda t} = 0$$

$$\overset{\parallel}{} \quad e^{\lambda t} \cdot \boxed{(\lambda^2 - 6\lambda - 7)}$$

$$\overset{e^{\lambda t} \neq 0}{\Longleftrightarrow} \quad \lambda^2 - 6\lambda - 7 = 0 \Rightarrow \lambda = -1, 7$$

$$\underset{(\lambda - 7)(\lambda + 1)}{\parallel}$$

$$\Rightarrow \boxed{y_1(t) = e^{-t}, \; y_2(t) = e^{7t}}$$

$\Rightarrow y(t) = C_1 \cdot e^{-t} + C_2 \cdot e^{7t}$ gen. solution.

# Linearity Principle for homogeneous linear differential equation

For $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$

If $y_1(t)$ and $y_2(t)$ are solutions,

then $y(t) = \underline{C_1} \cdot y_1(t) + C_2 \cdot y_2(t)$ is

a solution for any $C_1, C_2 \in \mathbb{R}$.

---

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0.$$

$$\Rightarrow \frac{d^2}{dt^2}(c_1 \cdot y_1) + p(t)\frac{d}{dt}(c_1 \cdot y_1) + q(t) \cdot c_1 y_1 = 0$$

$$\frac{d^2}{dt^2}(c_2 y_2) + p(t)\frac{d}{dt}(c_2 y_2) + q(t) c_2 y_2 = 0$$

$$\Rightarrow \frac{d^2}{dt^2}(c_1y_1+c_2y_2)+p(t)\frac{d}{dt}(c_1y_1+c_2y_2)$$
$$+q(t)\cdot(c_1y_1+c_2y_2)=0.$$

Method for: $\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=0$

2nd order linear
homogeneous, constant coefficients

constant.

$$y=e^{\lambda t} \rightsquigarrow \underline{\lambda^2+p\lambda+q=0}$$

$$\lambda=\frac{-p\pm\sqrt{p^2-4q}}{2}$$

- $\boxed{p^2-4q>0} \rightsquigarrow \lambda_1,\lambda_2 \rightsquigarrow \underset{\overset{\parallel}{y_1}}{e^{\lambda_1 t}}, \underset{\overset{\parallel}{y_2}}{e^{\lambda_2 t}}$

$\Rightarrow$ general solution $\underline{y(t)=c_1\cdot e^{\lambda_1 t}+c_2\cdot e^{\lambda_2 t}}$

- $p^2-4q=0$
- $p^2-4q<0$