Ex: $\frac{dy}{dt} = -4y + 9 \cdot e^{-4t}$

$\frac{9 \cdot e^{-1 \cdot t} \rightsquigarrow a \cdot e^{-t}}{9 \cdot e^{-4t}}$

step 1: $\frac{dy}{dt} = -4y \Rightarrow y_h(t) = C \cdot e^{-4t}$

step 2: Find a particular solution $\boxed{y_p(t) = a \cdot e^{-4t}}$

$\frac{dy}{dt} = a \cdot e^{-4t} \cdot (-4) \Rightarrow -4a = -4a + 9 \Rightarrow 0 = 9$

$\parallel \qquad \parallel$

$-4y + 9 \cdot e^{-4t} = -4 \cdot a \cdot e^{-4t} + 9 \cdot e^{-4t}$

Try: $y_p(t) = a \cdot t \cdot e^{-4t}$

$\frac{dy_p}{dt} = a \cdot e^{-4t} + a \cdot t \cdot e^{-4t} \cdot (-4) \Rightarrow a \cdot e^{-4t} = 9 \cdot e^{-4t}$

$\Downarrow$

$a = 9$

$\parallel$

$-4y + 9 \cdot e^{-4t} = -4 \cdot a \cdot t \cdot e^{-4t} + 9 \cdot e^{-4t}$

$\Rightarrow y_p(t) = 9 \cdot t \cdot e^{-4t}$

step 3: general solution:

$y(t) = y_h + y_p = C \cdot e^{-4t} + 9t \cdot e^{-4t}$.

Ex: $\frac{dy}{dt} = 2y - e^{\lambda \cdot t}$

step1: $\frac{dy}{dt} = 2y \Rightarrow y_h = C \cdot e^{2t}$

step2: 2 cases: • if $\lambda \neq 2$, use $y_p = a \cdot e^{\lambda t}$

• otherwise $\lambda = 2$, use $y_p = a \cdot t \cdot e^{\lambda t}$

$\rightarrow$ determine a

step3: $y(t) = C \cdot e^{2t} + y_p(t)$

Ex: $\boxed{\frac{dy}{dt} - 4y = \cos(2t)}$, $y(0) = -2$

step1: $\frac{dy}{dt} = 4y \Rightarrow y_h(t) = C \cdot e^{4t}$

step2: $\boxed{y_p(t) = a \cdot \cos(2t) + \boxed{b \cdot \sin(2t)}}$

$\boxed{\frac{dy_p}{dt} = -a \cdot \sin(2t) \cdot 2 + b \cdot \cos(2t) \cdot 2}$

$\frac{dy_p}{dt} - 4y_p = -a \cdot \sin(2t) \cdot 2 + b \cdot \cos(2t) \cdot 2$

$$-\left(4a\cdot\cos(2t)+4b\sin(2t)\right).$$

$$=(2b-4a)\cos(2t)+(-2a-4b)\cdot\sin(2t)$$

$$=1\cdot\cos(2t).$$

$$\begin{cases}2b-4a=1 \Rightarrow 2b-4\cdot(-2b)=10\cdot b=1 \Rightarrow b=\frac{1}{10}\\ 2a+4b=0 \Rightarrow a=-2b \Rightarrow a=-\frac{2}{10}=-\frac{1}{5}\end{cases}$$

$$\Rightarrow y_p(t)=-\frac{1}{5}\cos(2t)+\frac{1}{10}\sin(2t).$$

step 3: general solution

$$\boxed{y(t)=y_h+y_p=C\cdot e^{4t}+\left(-\frac{1}{5}\right)\cos(2t)+\frac{1}{10}\sin(2t)}$$

step 4: $y(0)=C\cdot e^{0}-\frac{1}{5}=-2 \Rightarrow C=-2+\frac{1}{5}=-\frac{9}{5}$

$$\Rightarrow y(t)=-\frac{9}{5}e^{4t}-\frac{1}{5}\cos(2t)+\frac{1}{10}\sin(2t).$$

$$\frac{dy}{dt} = 2y + \sin(5t). \qquad y_p = a\cdot\cos(5t) + b\cdot\sin(5t)$$

$$\frac{dy}{dt} = a(t)\,y + b(t)$$

$$\Rightarrow \quad \frac{dy}{dt} - a(t)y = b(t)$$

$$\frac{dy}{dt} + g(t)\,y = b(t)$$

Integration factor method:

$$\mu(t)\cdot\frac{dy}{dt} + \boxed{\mu(t)\,g(t)\,y} = b(t)\cdot\mu(t).$$

$$\boxed{\frac{d}{dt}\left(\underbrace{\mu(t)\cdot y}_{=z}\right) = b(t)\cdot\mu(t)}$$

$$= \mu\cdot\frac{dy}{dt} + \boxed{\frac{d\mu}{dt}\cdot y}$$

$$\Leftrightarrow \quad \frac{d\mu}{dt} = \mu\cdot g(t) \Rightarrow \underbrace{\frac{1}{\mu}d\mu}_{=d\cdot\log\mu} = g(t)\,dt$$

$$\Rightarrow \log\mu = \int g(t)\,dt \Rightarrow \boxed{\mu = e^{\int g(t)\,dt}}$$

integrating factor

$$\frac{dz}{dt} = \underline{b(t)\cdot\mu(t)} \Rightarrow \underset{\substack{\parallel \\ \mu(t)\cdot y(t)}}{z(t)} = \int b(t)\cdot\mu(t)\,dt$$

$$\Rightarrow \boxed{y(t) = (\mu(t))^{-1} \int b(t)\mu(t)\,dt}$$

Ex: $\frac{dy}{dt} = -\frac{y}{1+t} + t^2$

$$\frac{dy}{dt} + \underset{\substack{\parallel \\ g(t)}}{\left(\frac{1}{1+t}\right)} y = t^2 \qquad \int g(t)\,dt = \int \frac{1}{1+t}\,dt = \overset{\substack{\ln(1+t) \\ \parallel}}{\ln|1+t|}$$

$$\boxed{\mu(t) = e^{\int g(t)\,dt} = e^{\ln(1+t)} = 1+t}$$

$$\underset{\substack{\parallel \\ \frac{d}{dt}(\mu\cdot y)}}{(1+t)\frac{dy}{dt} + y} = \underset{\substack{\parallel \\ \frac{d}{dt}(\underset{\substack{\parallel \\ z}}{(1+t)y})}}{t^2(1+t)} \qquad \frac{dz}{dt} = t^2(1+t)$$

$$\Rightarrow z(t) = \int \overset{\substack{t^2+t^3 \\ \parallel}}{t^2(1+t)}\,dt$$

$$= \frac{1}{3}t^3 + \frac{1}{4}t^4 + \underline{C}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{1+t} z(t) = \frac{C}{1+t} + \frac{1}{1+t}\left(\frac{1}{3}t^3 + \frac{1}{4}t^4\right)}$$

is the general solution to the nonhomogeneous diff. eq.

Ex: $\underline{\dfrac{dy}{dt} - 4y} = \cos(2t).$

$$\mu(t) = e^{\int g(t)dt} = e^{\int -4\,dt} = e^{-4t}$$

$$\frac{d}{dt}\Big(\underbrace{e^{-4t} \cdot y}_{z(t)}\Big) = \cos(2t) \cdot e^{-4t}$$

$$\Rightarrow \quad z(t) = e^{-4t}y(t) = \boxed{\int \cos(2t) \cdot e^{-4t}\, dt}$$

$$= \int e^{-4t} \cdot d(\sin(2t)) \cdot \frac{1}{2}$$

$$= \frac{1}{2} e^{-4t} \cdot \sin(2t) - \frac{1}{2}\int \sin(2t) \cdot (-4) \cdot e^{-4t}\, dt$$

$$= \frac{1}{2} e^{-4t} \sin(2t) + 2 \cdot \underline{\int \sin(2t) \cdot e^{-4t}\, dt}$$

$$\int \sin(2t) \cdot e^{-4t}\, dt = \int e^{-4t} \cdot d(-\cos(2t)) \cdot \frac{1}{2}$$

$$= \frac{1}{2} e^{-4t} \sin(2t) - \Big(e^{-4t} \cdot \cos(2t) - \int \cos(2t) \cdot (-4) e^{-4t}\, dt\Big)$$

$$= \frac{1}{2} e^{-4t} \sin(2t) - e^{-4t}\cos(2t) - 4 \cdot \underline{\int \cos(2t) e^{-4t}\, dt}$$

$$\Rightarrow \quad \int \cos(2t) \cdot e^{-4t}\, dt = \frac{1}{5} \cdot \Big(\frac{1}{2} e^{-4t} \sin(2t) - e^{-4t} \cos(2t)\Big)$$

$$= \boxed{\frac{1}{10} e^{-4t} \sin(2t) - \frac{1}{5} e^{-4t} \cos(2t)}$$

$$\Rightarrow y_p(t) = \frac{1}{\mu} z(t) = \frac{1}{10} \sin(2t) - \frac{1}{5} \cos(2t)$$

$$y(t) = y_h + y_p$$
$$= C \cdot e^{4t} + \frac{1}{10}\sin(2t) - \frac{1}{5}\cos(2t)$$