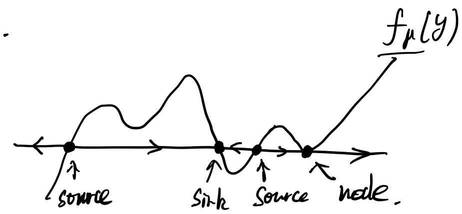
phase line and bifurcation diagram.

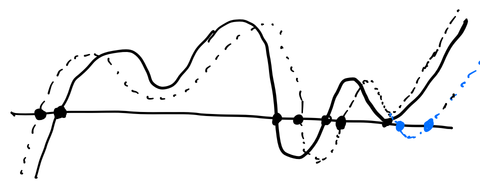
$$\frac{dy}{dt} = f_\mu(y) \longrightarrow \text{phase line}$$



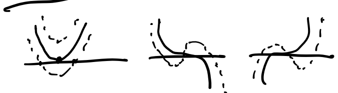
Assume y_0 is an equilibrium point

$$\Updownarrow$$

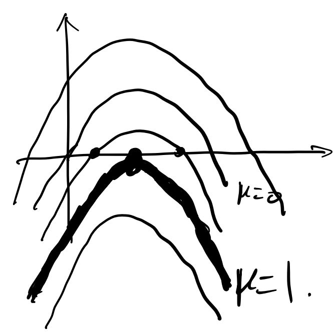
$$\boxed{f_\mu(y_0)=0}$$



- $f_\mu'(y_0)>0 \Rightarrow y_0$ is a source (stable i.e. can not be perturbed away).
- $f_\mu'(y_0)<0 \Rightarrow y_0$ is a sink (. - - —— - - .)
- $\boxed{f_\mu'(y_0)=0} \Rightarrow$ phase line can change near y_0 when we perturb μ



Ex: $\dfrac{dy}{dt} = 2y-y^2-\mu = f_\mu(y)$.

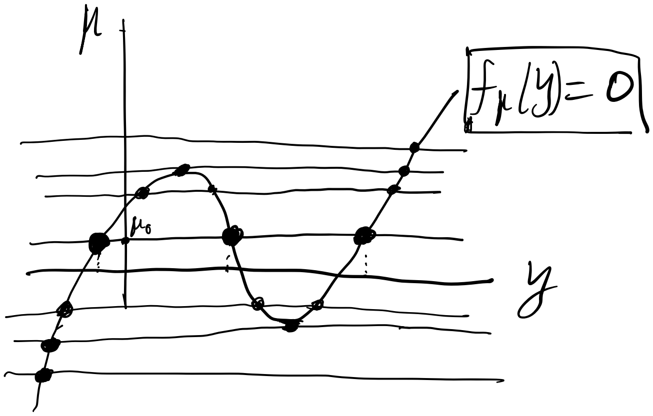


Q: how to find value of μ where bifurcation happens. ← if we perturb μ near μ_0, then the phase line changes.

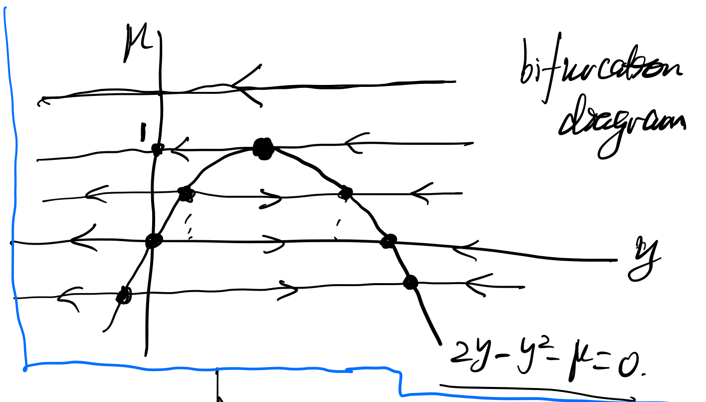
Solve: $\boxed{\begin{cases} f_\mu(y_0)=0 \\ f_\mu'(y_0)=0 \end{cases}}$ $\quad f_\mu' = \dfrac{d}{dy} f_\mu$

$$\begin{cases} 2y-y^2-\mu=0 \\ 2-2y=0 \end{cases} \Rightarrow \begin{array}{l} \boxed{\mu = 2y-y^2 = 2-1 = 1} \\ y=1 \end{array}$$

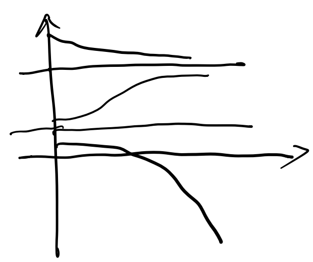
$$\frac{dy}{dt} = f_\mu(y)$$



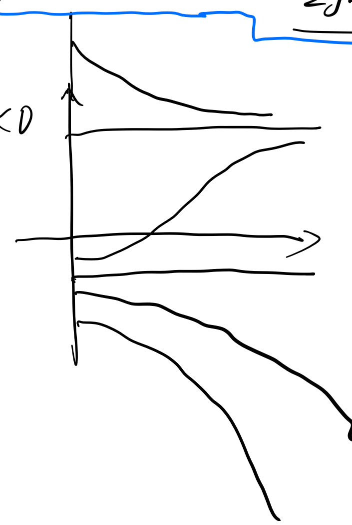
$$\frac{dy}{dt} = \underline{2y - y^2 - \mu} = 0$$



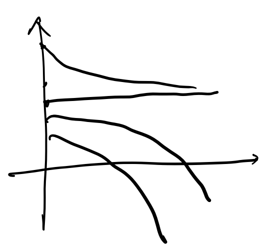
$\underline{0 < \mu < 1}$



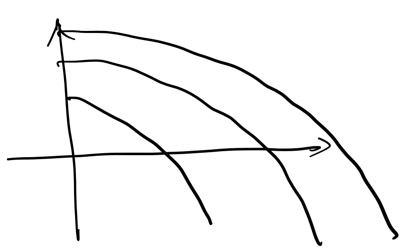
$\mu < 0$



$\underline{\mu = 1}$



$\underline{\mu > 1}$



Ex. $\dfrac{dy}{dt} = y^3 + \alpha y^2 = f_\alpha(y)$.

Find the value of α where the bifurcation happens.

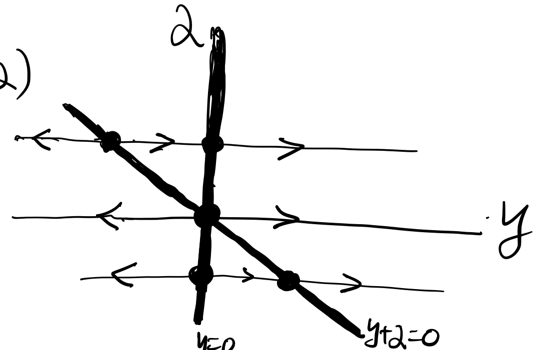
$$\begin{cases} f_\alpha(y)=0 \\ f_\alpha'(y)=0 \end{cases} \quad \begin{cases} y^3+\alpha y^2=0 = y^2\cdot(y+\alpha) \Rightarrow \boxed{y=0} \text{ or } y=-\alpha \\ \to 3y^2+2\alpha y=0 = y\cdot(3y+2\alpha) \end{cases}$$

$-\alpha\cdot(-3\alpha+2\alpha) = \alpha^2=0$

$\Rightarrow \alpha=0$

either $\boxed{y=0}$ or $y=-\alpha=0$

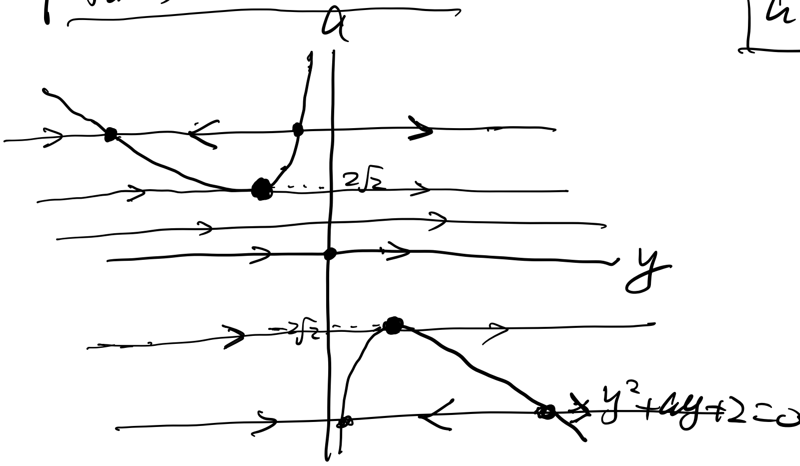
$f_\alpha(y)=0 = y^2(y+\alpha)$



Ex: $\dfrac{dy}{dt} = y^2 + ay + 2 = f_a(y)$. $\quad -y^2+2=0 \quad y^2=2 \Rightarrow y=\pm\sqrt{2}$.

$$\begin{cases} f_a(y) = y^2+ay+2=0 \Rightarrow y^2-2y\cdot y+2=0 \\ f_a'(y) = 2y+a=0 \Rightarrow a=-2y \Rightarrow a=-2\cdot(\pm\sqrt{2}) \end{cases}$$

$\boxed{a=\pm 2\sqrt{2}}$



$y^2+ay+2=0$

$a = -\dfrac{y^2+2}{y}$

$= -\left(y+\dfrac{2}{y}\right)$

linear equations

the associated homogeneous linear equation

$$\frac{dy}{dt} = \underbrace{a(t)\,y + b(t)}_{\uparrow} \Longrightarrow \underbrace{\frac{dy}{dt} = a(t)\cdot y}_{\uparrow \text{ separable}}$$

If $b(t) \neq 0$, then this is non-homogeneous.

Ex: $\underbrace{\frac{dy}{dt} = \sin(t)\cdot y + e^{-t}}_{\text{nonhomogeneous}} \rightsquigarrow \underbrace{\frac{dy}{dt} = \sin(t)\cdot y}_{\text{homogeneous}}$.

Linearity Principle:

For homogeneous linear equation $\frac{dy}{dt} = a(t)\cdot y$

If $y(t)$ is a solution, then $k\cdot y(t)$ is also a solution where k is a constant.

Pf: $\frac{dy}{dt} = a(t)\cdot y \Rightarrow k\frac{dy}{dt} = k\cdot a(t)\,y$

$$k\frac{dy}{dt} = \frac{d}{dt}(k\cdot y), \quad k\cdot a(t)\,y = a(t)(k\cdot y)$$

$$\frac{d}{dt}(k\cdot y) = a(t)(k\cdot y).$$

$$\frac{dy}{dt} = a(t)y + b(t) \Rightarrow k\cdot\frac{dy}{dt} = a(t)\cdot ky + \boxed{k\cdot b(t)}$$

$$k\cdot b(t) \neq b(t)$$

For nonhomogeneous linear equation

$$\boxed{\frac{dy}{dt} = a(t)\,y + b(t).}$$

$\rightsquigarrow$ associated homogeneous $\boxed{\frac{dy}{dt} = a(t)\cdot y}$

separable.

The general solution to the non-hom. linear eq. has the following form:

$$y(t) = \underbrace{k\cdot y_h(t)}_{\text{solution to the hom. linear eq.}} + \underbrace{y_p(t)}_{\text{particular solution to the non-hom. linear eq.}}$$

Ex: $\frac{dy}{dt} = -4y + 9e^{-t}$

step 1: $\frac{dy}{dt} = -4y \Rightarrow \int y^{-1}\,dy = -\int 4\,dt$

$$\Downarrow$$

$|y| = e^{c_1}\cdot e^{-4t} \Leftarrow \ln|y| = -4t + C_1$

$$\Downarrow$$

$y_h = C\cdot e^{-4t}$ is the general solution to $\frac{dy}{dt} = -4y$.

step 2: try $y_p(t) = a \cdot e^{-t}$

$\uparrow$ undetermined constant.

$$\frac{dy_p}{dt} = a \cdot e^{-t} \cdot (-1) = -a \cdot e^{-t}$$

$$\|$$

$$-4y + 9 \cdot e^{-t} = -4 \cdot a \cdot e^{-t} + 9 e^{-t}$$

$$\Rightarrow -a = -4a + 9 \Rightarrow 3a = 9 \Rightarrow a = 3$$

$$\Rightarrow y_p(t) = 3 \cdot e^{-t}.$$

step 3: general solution to non-hom. eq.

is $\boxed{y(t) = C \cdot e^{-4t} + 3 \cdot e^{-t}}$

$$\frac{dy}{dt} = a(t)\,y + b(t)$$

$$\frac{dy_p}{dt} = a(t)\,y_p + b(t)$$

$$\Rightarrow \frac{d(y-y_p)}{dt} = a(t)\cdot \underline{(y-y_p)}$$

↑ sol. to hom. linear eq.

$$\Rightarrow y - y_p = y_h \iff y = k\cdot \underline{y_h} + \underline{y_p}$$