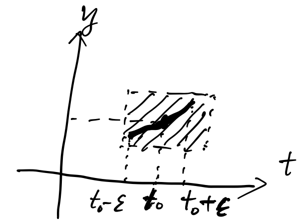
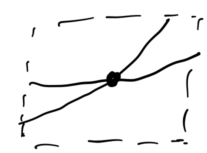
Existence and uniqueness for ODE (ordinary differential equations).

$$\frac{dy}{dt} = f(y,t)$$



Theorem: (existence) Suppose $f(y,t)$ is continuous near (t_0, y_0)

then there exists $\varepsilon > 0$, s.t. the solution exists on $(t_0-\varepsilon, t_0+\varepsilon)$ to the initial value problem $\frac{dy}{dt} = f(y,t)$, $y(t_0) = y_0$



Ex: $\frac{dy}{dt} = y^2$, $y(0) = 1$.

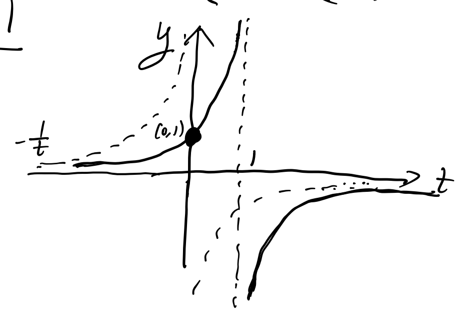
$$\int y^{-2}\,dy = \int dt = t + C_1 \Rightarrow -\frac{1}{y} = t + C_1 \xrightarrow[y=1]{t=0} -1 = 0 + C_1 \Rightarrow C_1 = -1$$

$$\frac{1}{-2+1} y^{-1} = -y^{-1}$$

$$\Rightarrow -\frac{1}{y} = t - 1 \Rightarrow y = \frac{1}{1-t} = -\frac{1}{t-1}$$

$$\left(y' = \frac{1}{(1-t)^2} = y^2\right)$$

$y = \frac{1}{1-t}$ exists for $-\infty < t < 1$



defined only for $y \geq 0$

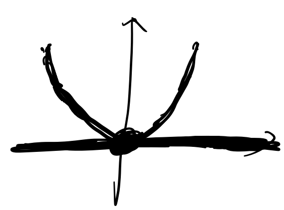
Ex: $\frac{dy}{dt} = y^{\frac{1}{2}}$ $y(0) = 0$

$$\int y^{-\frac{1}{2}}\,dy = \int dt = t + C \Rightarrow 2y^{\frac{1}{2}} = t + C \xrightarrow[y=0]{t=0} 2 \cdot 0 = 0 + C$$

$$\Rightarrow C = 0$$

$$\frac{1}{-\frac{1}{2}+1} y^{-\frac{1}{2}+1} = 2 \cdot y^{\frac{1}{2}}$$

$$\Rightarrow 2y^{\frac{1}{2}} = t \Rightarrow y = \left(\frac{t}{2}\right)^2 = \frac{t^2}{4}$$



Ex: $\frac{dy}{dt} = y^{\frac{1}{3}}$. $y(0) = 0$.

$$\int y^{-\frac{1}{3}}\, dy = \int dt = t + C \Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t + C$$

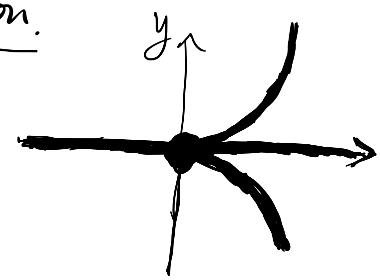
$$0 = 0 + C \Rightarrow C = 0$$

$$\frac{1}{-\frac{1}{3}+1} y^{-\frac{1}{3}+1} = \frac{3}{2} \cdot y^{\frac{2}{3}} \qquad \Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t$$

$$\Rightarrow \boxed{y = \left(\frac{2t}{3}\right)^{\frac{3}{2}}} \quad t \geq 0.$$

nonzero solution

$y = 0$ is the equilibrium solution.



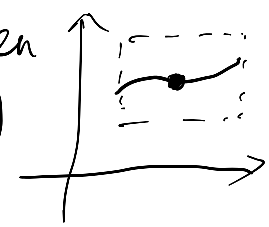
$$\frac{3}{2} y^{\frac{2}{3}} = t \Rightarrow y^{\frac{2}{3}} = \frac{2t}{3}$$

$$\Rightarrow y = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$$

solutions: not unique

- $y(t) \equiv 0$
- $y(t) = \begin{cases} 0 & t \leq 0 \\ \left(\frac{2}{3}t\right)^{\frac{3}{2}} & t > 0 \end{cases}$
- $y(t) = \begin{cases} 0 & \\ -\left(\frac{2}{3}t\right)^{\frac{3}{2}} & t > 0. \end{cases}$

Thm : (Uniqueness) Suppose both $\underline{f(t,y)}$ and $\frac{\partial f}{\partial y}(t,y)$ are $\underline{\text{continuous}}$ near (t_0, y_0). Then the solution passing through (t_0, y_0) is unique.



If $\boxed{f(t,y) = y^{\alpha}}$, $\alpha > 0$. then f is continuous

but $\frac{\partial f}{\partial y} = \alpha \cdot \underline{y^{\alpha-1}}$ is continuous near $(0,0)$ only when $\alpha \geq 1$. When $\alpha < 1$, the solution to $\begin{cases} \frac{dy}{dt} = y^{\alpha} \\ y(0) = 1 \end{cases}$ is not unique.

Application of uniqueness.

- $$\frac{dP}{dt} = \underline{k \cdot P\left(1 - \frac{P}{N}\right)}.$$

$\uparrow$ Smooth in P

- compare solutions

$$\frac{dy}{dt} = y^2 \rightsquigarrow -\frac{1}{y} = t + C_1$$

$$\Rightarrow y = -\frac{1}{t+C_1} = -\frac{1}{t-C_2}$$

$$\underline{y = -\frac{1}{t-1}}, \quad \underline{y = -\frac{1}{t-2}}$$