$$\frac{dy}{dt} = t - y^2$$

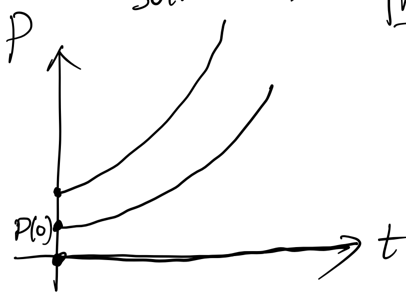
analytic

qualitative

numerical

- Exponential growth model : $\boxed{\frac{dP}{dt} = k \cdot P}$ constant parameter.

solutions : $\boxed{P = P(t) = C \cdot e^{kt}}$ for any constant $C \in \mathbb{R}$



$P(0) = C \cdot e^{k \cdot 0} = C$

initial population.

- Separable differential equation : $\boxed{\frac{dy}{dt} = f(t, y) = \underline{g(t)} \cdot \underline{h(y)}}$

$$\frac{dy}{dt} = g(t) \cdot h(y) \longrightarrow \frac{dy}{h(y)} = g(t) \cdot dt$$

$$\left(A'(y) = \frac{1}{h(y)}\right) \int \frac{dy}{h(y)} = \int g(t)\, dt$$

integ. constant

$$\boxed{y = y(t, C)} \Longleftarrow \boxed{A(y) = B(t) + C}$$

$$\frac{dP}{dt} = k\cdot P \longrightarrow \frac{dP}{P} = k\cdot dt. \quad (P \neq 0)$$

$$\downarrow$$

$$\ln|P| = \int \frac{dP}{P} = \int k\, dt = k\cdot t + C_1$$

$\cancel{+C_1'}$ $\qquad$ $\cancel{C_1 - C_1'}$

$$\downarrow$$

$$\begin{cases} P, & P \geq 0 \\ -P, & P < 0 \end{cases} = |P| = e^{\ln|P|} = e^{kt + C_1} = e^{C_1}\cdot e^{kt}$$

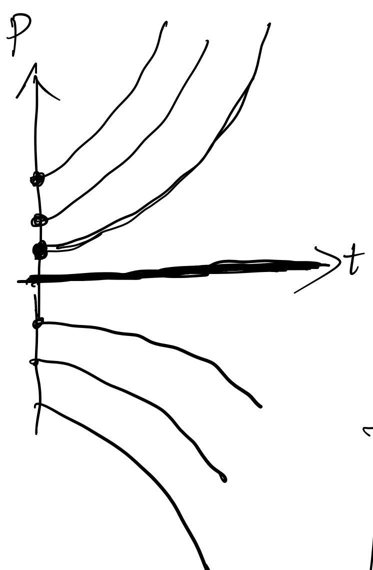
$$\| \qquad$$

$$C_2 \cdot e^{kt}. \quad C_2 > 0$$

$$\downarrow$$

$$P(t) = C_2\cdot e^{kt} \text{ or } P(t) = -C_2 \cdot e^{kt}.$$

$$\downarrow$$



$$\boxed{P(t) = C\cdot e^{kt}, \quad \text{for } C \in \mathbb{R}}$$

$\|$ C_2 or $-C_2$ $\qquad$ $\neq 0$ $\qquad$ set of real numbers

$$\boxed{\begin{array}{l} P(t) = C\cdot e^{kt} \quad \text{for } C \in \mathbb{R} \\ \text{general solution to } \frac{dP}{dt} = k\cdot P \end{array}}$$

$$\begin{cases} \frac{dP}{dt} = k\cdot P \\ P(0) = P_0 \end{cases} \rightarrow P(t) = C\cdot e^{kt}$$

$$P(0) = C\cdot e^0 = C = P_0 \Rightarrow \boxed{P(t) = P_0 \cdot e^{kt}}$$

the solution to the IVP

IVP (initial value problem)

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t}{y - t^2 \cdot y} = \dfrac{t}{1-t^2} \cdot \dfrac{1}{y} \\ y(0) = 4 \end{cases}$$

$$\int y\,dy = \int \frac{t}{1-t^2}\,dt \xlongequal[t\,dt = -\frac{du}{2}]{\substack{u = 1-t^2 \\ du = -2t\,dt}} \int -\frac{du}{2u}$$

$$\frac{1}{2}y^2 \qquad\qquad -\frac{1}{2}\ln|u| + C_1$$

$$C_1' + \frac{1}{2}y^2 = -\frac{1}{2}\ln|1-t^2| + C_1 - C_1'$$

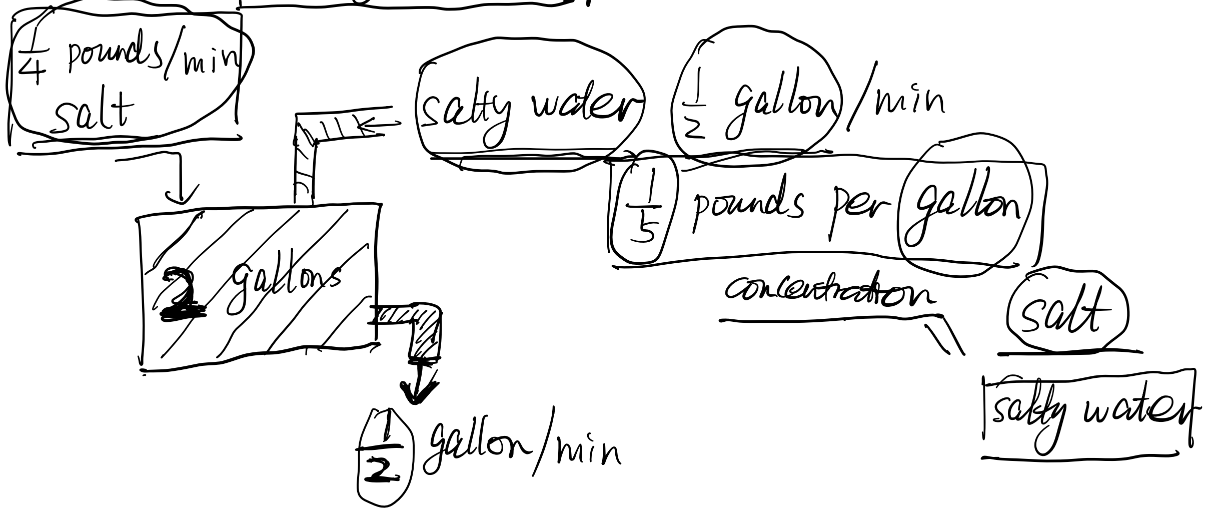
$$y^2 = -\ln|1-t^2| + C_2$$

general solution: $y = \sqrt{-\ln|1-t^2| + C_2}$

$$y(0) = \sqrt{-\ln 1 + C_2} = \sqrt{C_2} = 4 \Rightarrow C_2 = 16$$

$y(t) = \sqrt{16 - \ln|1-t^2|}$ solution to the IVP.

Mixing Problem



$y(0) = 1$ pound of salt in the tank

Q: what is the amount of salt in the tank at time t?

$\boxed{y: \text{ amount of salt}}$

$$\frac{dy}{dt} = \frac{1}{4} + \boxed{\frac{1}{2} \times \frac{1}{5}} - \boxed{\frac{1}{2} \times \frac{y}{2}}$$

$\frac{1}{4} + \frac{1}{10} = \frac{5+2}{20} = \frac{7}{20}$

change rate

$$= \frac{7}{20} - \frac{y}{4} = \underbrace{\frac{1}{20}(7-5y)}_{\text{autonomous}} = \frac{dy}{dt}$$

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{20}(7-5y) & (*) \\ y(0) = 1 \end{cases}$$

$$\int \frac{1}{7-5y}\,dy = \int \frac{1}{20}\,dt = \frac{1}{20}t + C_1 \qquad \boxed{7-5y \neq 0}$$

$y = \frac{7}{5}$ is a sol.

$$\parallel$$

$$-\frac{1}{5}\int \frac{d(7-5y)}{7-5y} = -\frac{1}{5}\ln|7-5y| = \frac{1}{20}t + C_1$$

$$\ln|7-5y| = -\frac{1}{4}t - 5C_1 = -\frac{1}{4}t + C_2$$

$$\begin{cases} 7-5y, & y<\frac{7}{5} \\ 5y-7, & y>\frac{7}{5} \end{cases} = |7-5y| = e^{C_2}\cdot e^{-\frac{1}{4}t} = C_3\cdot e^{-\frac{1}{4}t}$$

$C_3 = e^{C_2} > 0.$

$$\begin{cases} 7-5y = C_3\cdot e^{-\frac{1}{4}t}, & y<\frac{7}{5} \\ \text{or } 5y-7 = C_3\cdot e^{-\frac{1}{4}t} & y>\frac{7}{5} \end{cases}$$

$$\Rightarrow \boxed{7-5y = C\cdot e^{-\frac{1}{4}t}, \quad C\in\mathbb{R}.}$$

$$\Rightarrow y = \frac{1}{5}\left(7 - C\cdot e^{-\frac{1}{4}t}\right). \quad C\in\mathbb{R}$$

general sol. to (*)

$$y(0) = \frac{1}{5}(7 - C\cdot e^{0}) = \frac{1}{5}(7-C) = 1 \Rightarrow C = 2.$$

$$\Rightarrow \boxed{y(t) = \frac{1}{5}\left(7 - 2\cdot e^{-\frac{1}{4}t}\right)}$$

$$\lim_{t\to+\infty} y(t) = \frac{7}{5} = 1.4 \text{ pounds.}$$

$$\frac{dy}{dt} = \frac{1}{20}(7-5y)$$

$y = 1.4$ is the equilibrium

$$\frac{dy}{dt} = f(y) = f(y) \cdot 1 \rightarrow \frac{dy}{f(y)} = dt.$$