- dot product, orthogonal vectors
- cross product, (unit) normal vector : $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$
- Equations for lines and planes ; circles, spheres
- quadric surfaces (paraboloids, cones, spheres, . . .)

find the center and the radius by completing the square

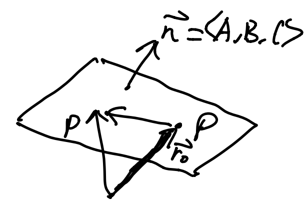
- cylinders, graph $z = f(x,y)$
- partial derivatives, higher order derivatives
- Lagrange multiplier method
- Triple integrals in rectangular, cylindrical coordinates (find correct limits)

$dV = dz\, r\, dr\, d\theta$

- Surface parametrization, surface integrals of functions / vector fields
- Green's theorem: both circulation and flux forms
- Stokes' theorem, curl of vector fields
- divergence theorem

<u>Ex:</u> Find the point on the plane $x + 5y + 3z = 18$ closest to the point $(1,1,1)$.

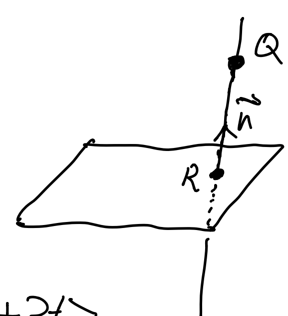
$\vec{n} = \langle 1, 5, 3 \rangle$



$$\text{plane} = \{ P \mid \overrightarrow{P_0P} \perp \vec{n} \}$$

$$\Rightarrow \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

$$A \cdot (x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



parameter equation for <u>line</u>: $\vec{r} = \vec{r_Q} + t \cdot \vec{n}$

$$\vec{r}(t) = \langle 1,1,1 \rangle + t \cdot \langle 1, 5, 3 \rangle = \langle 1+t, 1+5t, 1+3t \rangle$$

Find R: $\underbrace{(1+t)+5\cdot(1+5t)+3\cdot(1+3t)}_{(1+25+9)t+(1+5+3)=35t+9}=18 \Rightarrow 35t=9$

$\Downarrow$

$t=\frac{9}{35}$

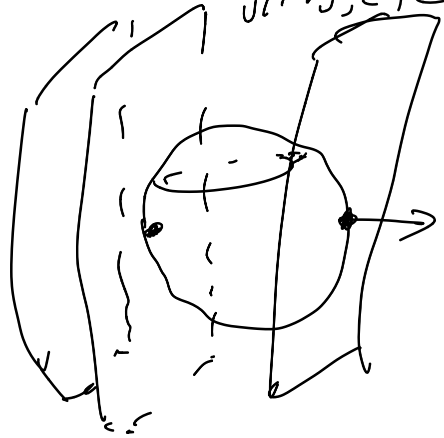
$R=\left(1+\frac{9}{35},\ 1+5\cdot\frac{9}{35},\ 1+3\cdot\frac{9}{35}\right)$

$=\left(\frac{44}{35},\ 1+\frac{9}{7}=\frac{16}{7},\ \frac{35+27}{35}=\frac{62}{35}\right)$

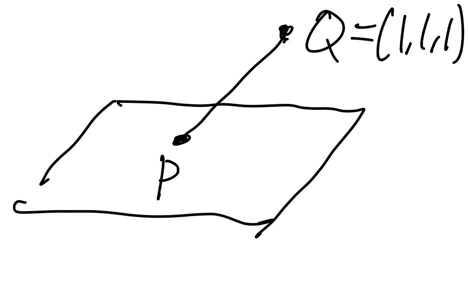
Lagrange multiplier method.

minimum or maximum of $f(x,y,z)$ under the constraint $g(x,y,z)=C$.

$$\begin{cases} \nabla f=\lambda\cdot\nabla g \\ g=C. \end{cases}$$



$$\Leftrightarrow \begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_y \\ f_z=\lambda g_z \\ g=C. \end{cases} \longrightarrow \text{Solve to get } (x,y,z,\lambda).$$



Minimize $(\text{distance of } \overrightarrow{PQ})^2$

$$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2.$$

Constraint : $x + 5y + 3z = 18$

$$\underset{g(x,y,z)}{\shortparallel}$$

$$t = \frac{\lambda}{2}$$

$$\begin{cases} 2(x-1) = \lambda \cdot 1 & \Rightarrow x = \frac{\lambda}{2} + 1 = \underline{1+t} \\ 2(y-1) = \lambda \cdot 5 & \Rightarrow y = 5\frac{\lambda}{2} + 1 = \overline{1+5t} \\ 2(z-1) = \lambda \cdot 3 & \Rightarrow z = 3 \cdot \frac{\lambda}{2} + 1 = \underline{1+3t} \\ \underline{x + 5y + 3z = 18} \end{cases}$$

$$\text{dist} = f^{\frac{1}{2}} = \sqrt{t^2 + (5t)^2 + (3t)^2} = t \cdot \sqrt{1+25+9} = \sqrt{35}\, t$$

from

Q to the plane

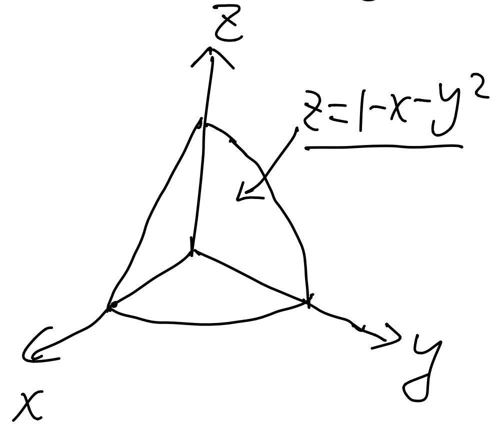
$$\shortparallel$$

$$\sqrt{35} \cdot \frac{9}{35} = \frac{9}{\sqrt{35}}$$

Ex: Region D bounded by $\boxed{z = 1 - x - y^2}$, $\boxed{x=0, \quad y=0, \quad z=0}$.

$$\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$$

Find the flux of $\vec{F}$ across the boundary of D.



when $x=0$: $z = 1 - y^2$

when $y=0$: $z = 1 - x$

when $z=0$: $x = 1 - y^2$

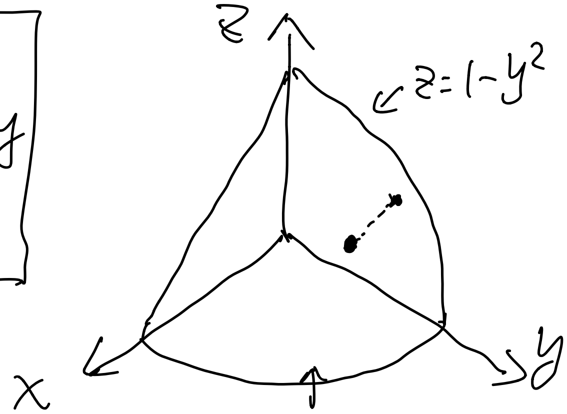
Using the divergence theorem: $\quad \mathrm{div}(\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

$$\iint_S \vec{F} \cdot \vec{n} \, dA = \iiint_D (\nabla \cdot \vec{F}) \, dV$$

$$\mathrm{div}(\vec{F}) = \frac{\partial y}{\partial x} + \frac{\partial (-x)}{\partial y} + \frac{\partial z}{\partial z} = 1 \qquad \boxed{\iiint_D dV}$$

Write $\iiint_D dV = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} dx \, dz \, dy$

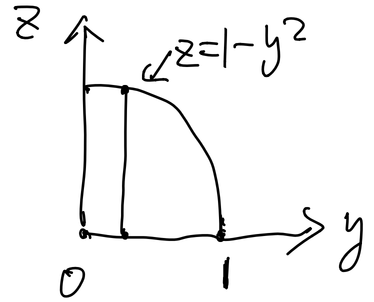
$$\boxed{\int_0^1 \int_0^{1-y^2} \left(\int_0^{1-y^2-z} dx \right) dz \; dy}$$



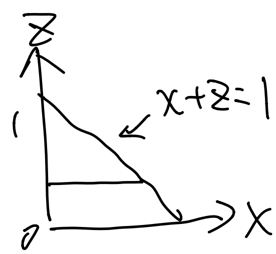
$z = 1-x-y^2$

$\Updownarrow$

$x = 1-y^2-z$



$$\int_0^1 \int_0^{1-z} \int_0^{\sqrt{1-x-z}} \underline{dy}\; dx\; dz$$



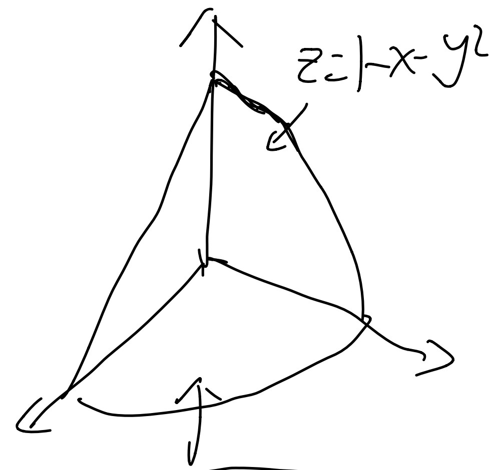
$y^2 = 1-x-z \Leftarrow z = 1-x-y^2$

---

flux of $\vec{F}$ across the surface $z=1-x-y^2$.
in the 1st octant.

write down the surface integral that calculates

$\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$



$\vec{r} = \vec{r}(u,v)$

$\iint_S \vec{F} \cdot \underbrace{\vec{n}}_{\text{unit normal}} \, dA$

$=$

$\iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|} \, du\,dv$

$=$

$\iint_{\underset{uv\text{-plane}}{D}} \boxed{\vec{F} \cdot \vec{r}_u \times \vec{r}_v} \, du\,dv$

$\vec{r}(x,y) = \langle \underset{u}{x}, \underset{v}{y}, \underset{f(x,y)=z}{1-x-y^2} \rangle$

$\vec{r}_x = \langle 1, 0, -1 \rangle$

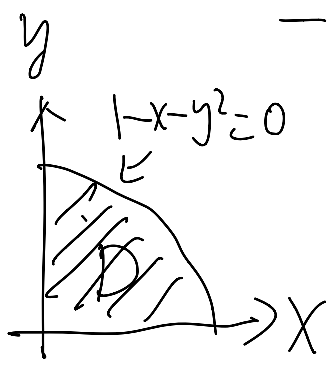
$$\vec{r}_y = \langle 0, 1, -2y \rangle$$

$$\underline{\vec{F} \cdot \vec{r}_u \times \vec{r}_v} = \begin{vmatrix} y & -x & z \\ 1 & 0 & -1 \\ 0 & 1 & -2y \end{vmatrix}$$

$$\vec{F} \cdot \vec{r}_x \times \vec{r}_y$$

$$= -1 \cdot \begin{vmatrix} y & z \\ 1 & -1 \end{vmatrix} + (-2y) \cdot \begin{vmatrix} y & -x \\ 1 & 0 \end{vmatrix}$$

$$= -1 \cdot (-y - z) - 2y \cdot (x)$$



$$= y + z - 2xy \Big|_{z = 1 - x - y^2}$$

$$= y + 1 - x - y^2 - 2xy$$

$$\rightarrow \underline{\iint_D (y + 1 - x - y^2 - 2xy)\, dx\, dy}$$

divergence thm: $\iint_S \vec{F}\cdot\vec{n} = \iiint_D \nabla\cdot\vec{F}\, dV$

surface with no bdry

Stokes' thm: $\iint_S \text{curl}\,\vec{F}\cdot\vec{n} = \oint_C \vec{F}\cdot d\vec{r}$

surface with a boundary curve C

$\square\left(\left(\frac{u}{v}\right)^n + uv^m\right)$

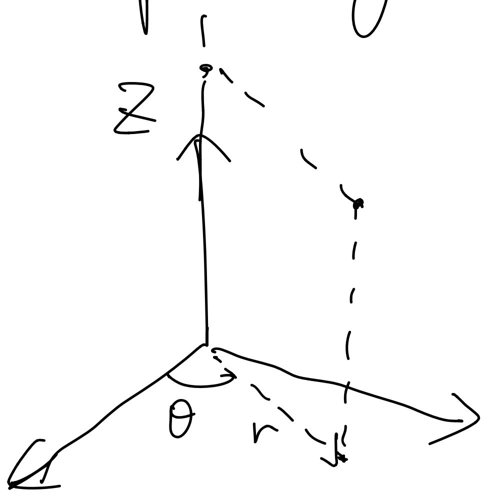
$\frac{2u}{v}\cdot\left(\left(\frac{u}{v}\right)^2 + uv^3\right)$ X

$$\frac{2u}{v}\left(\left(\frac{u}{v}\right)^2 + (uv)^2\right)v$$

$$\|$$

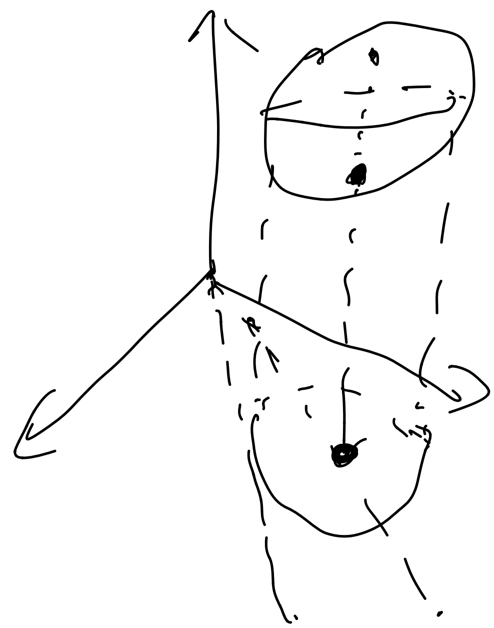
$$2\left(\left(\frac{u}{v}\right)^3 + u^3 v\right)$$

Triple integrals in cylindrical/spherical coordinates.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint_D f(x,y,z)\,\underline{dV} = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta$$



Ex: Solid region inside $x^2+y^2+z^2=4$

cut out by $x^2+y^2=2y$
(inside)

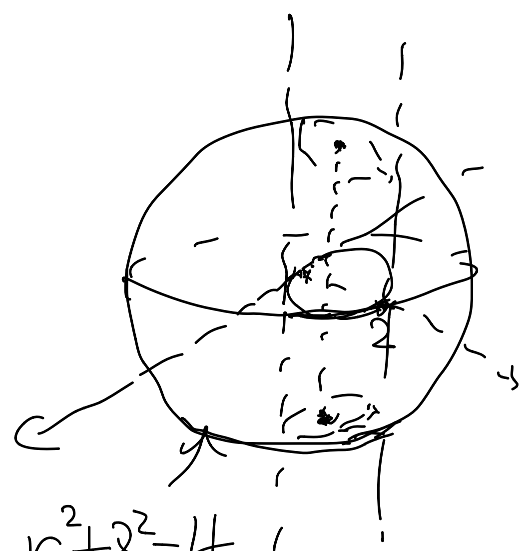
$x^2+y^2-2y=0$

$\Updownarrow$

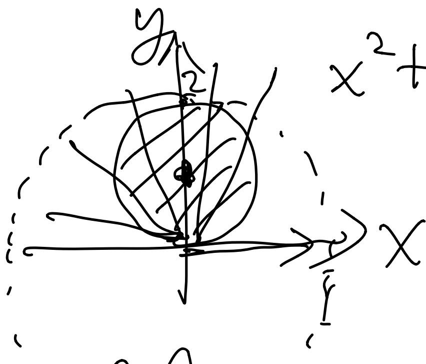
$x^2+y^2-2y+1=1$

$\shortparallel$

$x^2+(y-1)^2=1$



$r^2+z^2=4$



$x^2+y^2-2y = r^2-2\cdot r\cdot\sin\theta=0$

$\Leftrightarrow r = 2\cdot\sin\theta$

$$\iiint_D dV = \int_0^{\pi}\int_0^{2\cdot\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$$

$f(x,y,z)$

$f(r\cdot\cos\theta, r\sin\theta, z)$