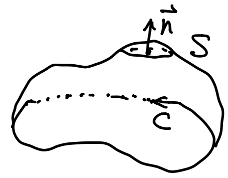
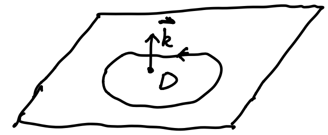
Stokes' theorem



$$\vec F = M\vec i + N\vec j + P\vec k$$

$$\oint_C \vec F \cdot d\vec r = \iint_S \text{curl}\,\vec F \cdot \vec n \, dA$$



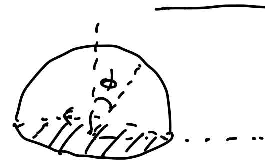
$$\oint_C \vec F \cdot d\vec r = \iint_D \text{curl}\,\vec F \cdot \vec k \, dx\,dy$$

$$\text{curl}\,\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec i \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \vec j \cdot \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \vec k \cdot \underline{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$$

Ex: $\underline{\vec F = 2y\,\vec i + (5-2x)\vec j + (z^2-2)\vec k}$.

$S:\ \vec r(\phi,\theta) = (\sqrt3 \sin\phi \cos\theta)\vec i + (\sqrt3 \sin\phi \sin\theta)\vec j + (\sqrt3 \cos\phi)\vec k$ $\quad 0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$

$$\underline{\oint_C \vec F \cdot d\vec r} = \boxed{\iint_S (\text{curl}\,\vec F \cdot \vec n)\, dA}$$



$$= \underline{\iint_{S_1} (\text{curl}\,\vec F \cdot \vec k)\, dx\,dy}.$$

$C:\ \vec r(\theta) = \langle \sqrt3 \cos\theta, \sqrt3 \sin\theta, 0 \rangle$

$\underline{\vec r\,'(\theta) = \langle -\sqrt3 \sin\theta, \sqrt3\cos\theta, 0\rangle}$

$$\oint_C \vec F \cdot d\vec r = \int_0^{2\pi} \vec F \cdot \vec r\,'(\theta)\, d\theta = \int_0^{2\pi} (-6 + 5\sqrt3 \cos\theta)\, d\theta = -6 \times 2\pi = \underline{-12\pi}.$$

$$\vec F \cdot \vec r\,'(\theta) = \left(2y \cdot (-\sqrt3 \sin\theta) + (5-2x)\cdot \sqrt3 \cos\theta\right)\Big|_{(x,y)=(\sqrt3\cos\theta,\, \sqrt3 \sin\theta)}$$

$$= 2\sqrt3 \sin\theta \cdot (-\sqrt3 \sin\theta) + (5 - 2\cdot\sqrt3\cos\theta)\cdot\sqrt3\cos\theta$$

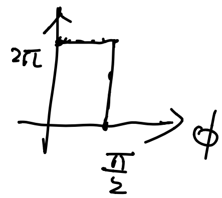
$$= -6\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1} + 5\sqrt3 \cos\theta$$

$$\text{curl}\,\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 5-2x & z^2-2 \end{vmatrix} = \vec i \cdot 0 + \vec j \cdot 0 + \vec k \cdot \underbrace{\left(\frac{\partial}{\partial x}(5-2x) - \frac{\partial}{\partial y}(2y)\right)}_{-2-2=-4.}$$

$$\iint_{S_1} \text{curl}\,\vec{F}\cdot\vec{n}\,dx\,dy = \iint_{\circ} (-4)\,dx\,dy = -4\pi(\sqrt{3})^2$$
$$= -12\pi$$

$$\vec{r}(\phi,\theta) = \langle \sqrt{3}\sin\phi\cos\theta,\ \sqrt{3}\sin\phi\sin\theta,\ \sqrt{3}\cos\phi \rangle$$

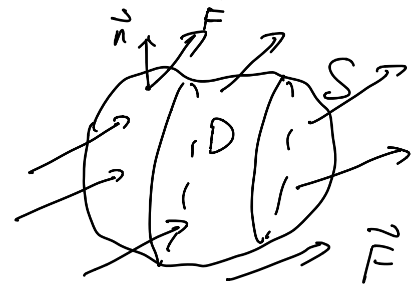
$$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA$$
$$\parallel$$
$$\text{curl}\,\vec{F}\cdot\frac{\vec{r}_\phi\times\vec{r}_\theta}{\cancel{|\vec{r}_\phi\times\vec{r}_\theta|}}\ \cancel{|\vec{r}_\phi\times\vec{r}_\theta|}\ d\phi\,d\theta$$



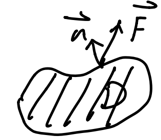
$$\parallel$$
$$\iint_S \left(\text{curl}\,\vec{F}\cdot\vec{r}_\phi\times\vec{r}_\theta\right)d\phi\,d\theta = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}(-6\sin(2\phi))\,d\phi\right)d\theta \qquad -12\pi \checkmark$$
$$\parallel \qquad\qquad \parallel$$
$$2\pi\cdot\frac{6}{2}\cdot\cos(2\phi)\Big|_0^{\frac{\pi}{2}} = 6\pi\cdot(-1-1)$$

$$\begin{vmatrix} 0 & 0 & -4 \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix} = (-4)\cdot 3\cdot\sin\phi\cos\phi\cdot(\cos^2\theta+\sin^2\theta)$$
$$= -12\cdot\sin\phi\cos\phi = -6\cdot\sin(2\phi)$$

Divergence Theorem



$$\iint_S \vec{F}\cdot\vec{n}\,dA$$
$$= \iiint_D (\text{div}\,\vec{F})\,dV$$



$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{div}\,\vec{F}\,dx\,dy$$

Green's Theorem.

$\vec F = M\vec i + N\vec j + P\vec k$, $\operatorname{div}\vec F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$ scalar function

flux density $= \underbrace{\nabla \cdot \vec F}_{\parallel}$

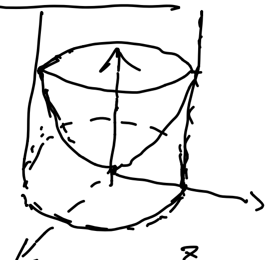
$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle$

$$\boxed{\nabla \times \vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \quad \text{vector field.}}$$

Ex: $\vec F = 5y\,\vec i + 5xy\,\vec j - 8z\,\vec k$

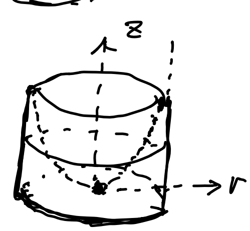
Calculate the flux of $\vec F$ across the boundary of D where D is the region inside the cylinder $x^2+\overset{r^2}{\overset{\parallel}{y^2}} \le 4$ between $z=0$ and the paraboloid $z = x^2+y^2 = r^2$

$\iint_S \vec F \cdot \vec n \, dA = \iiint_D \operatorname{div}\vec F \, dV$



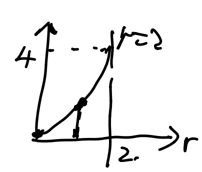
$\operatorname{div}\vec F = \nabla \cdot \vec F = \frac{\partial}{\partial x}(5y) + \frac{\partial}{\partial y}(5xy) + \frac{\partial}{\partial z}(-8z)$

$= 5x - 8$



$\iiint_D (5x-8)\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} (5r\cos\theta - 8)\,dz\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (5r^3\cos\theta - 8r^2)\,r\,dr\,d\theta$



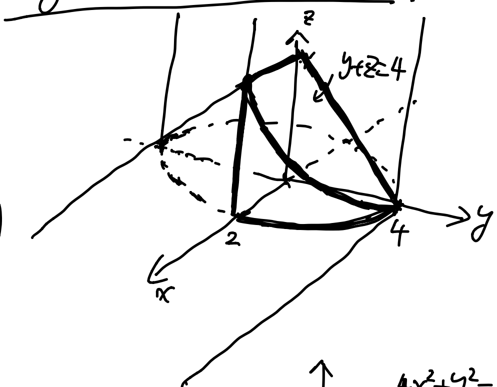
$= \underbrace{\left(\int_0^{2\pi}\cos\theta\,d\theta\right) \cdot \left(\int dr\right)}_{\sin\theta|_0^{2\pi}} - \int_0^{2\pi} d\theta \cdot \int_0^2 (+8r^3\,dr)$

$= -2\pi \cdot \frac{8}{4} r^4 \Big|_0^2 = -2\pi \times 2 \times \underbrace{2^4}_{16} = -64\pi.$

Ex: $\vec{F} = 8xz\,\vec{i} - 6xy\,\vec{j} - 4z^2\,\vec{k}$

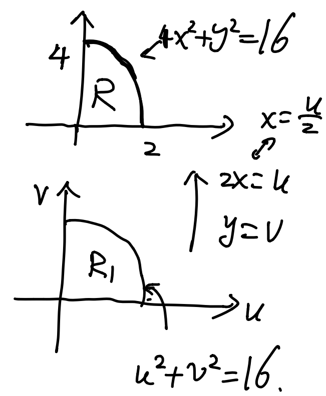
boundary of the region D: the wedge cut from the first Octant by the plane $y+z=4$ and the elliptical cylinder $4x^2+y^2=16$.

$$\boxed{\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_D \nabla\cdot\vec{F}\,dV}$$



$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(8xz)+\frac{\partial}{\partial y}(-6xy)+\frac{\partial}{\partial z}(-4z^2)$$

$$\operatorname{div}\vec{F} = 8z-6x-8z = \underline{-6x}.$$



$$\iiint_D \operatorname{div}\vec{F}\,dV = \iint_R \int_0^{4-y} (-6x)\,dz\,dx\,dy$$

$$= \iint_R -6x(4-y)\,dx\,dy$$

$$= \iint_{R_1} -6\cdot\frac{u}{2}\cdot(4-v)\underbrace{\frac{\partial(x,y)}{\partial(u,v)}}_{\left|\begin{smallmatrix}\frac12 & \\ & 1\end{smallmatrix}\right| = \frac12}\,du\,dv$$

$$= -\frac{6}{2}\cdot\frac{1}{2}\cdot\iint_{\{u^2+v^2\le 16,\ u\ge 0,\ v\ge 0\}} u\cdot(4-v)\,du\,dv \qquad r^2 = u^2+v^2$$

$$= -\frac{3}{2}\cdot\int_0^{\frac{\pi}{2}}\int_0^4 \underbrace{r\cos\theta\cdot(4-r\sin\theta)\,r}_{(4r^2\cos\theta - r^3\cos\theta\sin\theta)}\,dr\,d\theta$$

$$= -\frac{3}{2}\cdot\left(\int_0^{\frac{\pi}{2}}\cos\theta\cdot\int_0^4 4r^2\,dr - \int_0^{\frac{\pi}{2}}\frac{\sin(2\theta)}{2}\,d\theta\cdot\int_0^4 r^3\,dr\right)$$

$$= -\frac{3}{2}\cdot\left[\left(\sin\theta\Big|_0^{\frac{\pi}{2}}\right)\cdot\frac{4}{3}r^3\Big|_0^4 + \frac{1}{2}\cdot\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}}\cdot\frac{1}{4}\cdot r^4\Big|_0^4\right]$$

$$= -\frac{3}{2}\left[1\cdot\ \frac{4}{8}\cdot 4^3 + \frac{1}{4}\cdot(-1-1)\cdot\frac{1}{4}\cdot 4^4\right]$$

$$= -\frac{3}{2}\cdot\left[\frac{4^4}{3} - 2\times 4^2\right] = -2\times 4^3 + 3\times 4^2$$

$$= -2\times 64 + 3\times 16$$

$$= -128 + 48 = -80$$