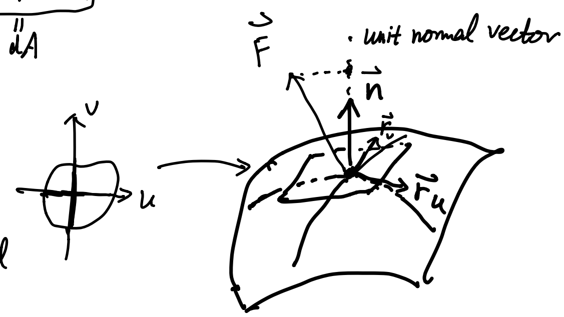
surface integrals $\quad \vec r = \vec r(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$

$$\iint_S f(x,y,z)\,dA = \iint_D f(\vec r(u,v))\,\underbrace{|\vec r_u \times \vec r_v|\,du\,dv}_{dA}$$

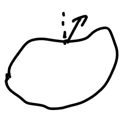
$\vec F = M\vec i + N\vec j + P\vec k$ vector field



$$\iint_S \vec F\cdot\vec n\,dA$$

flux of $\vec F$ across S in the direction defined by $\vec n$

$\left(\text{flux across a curve: } \int_C \vec F\cdot\vec n\,ds\right)$



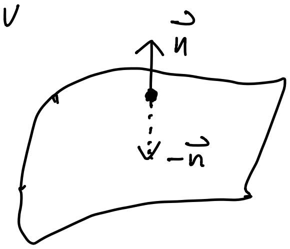
$\vec n \perp \vec r_u, \quad \vec n \perp \vec r_v$

$$\vec n = \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|}$$

$$\iint_S \vec F\cdot\vec n\,dA = \iint_D \vec F\cdot\frac{\vec r_u\times\vec r_v}{\cancel{|\vec r_u\times\vec r_v|}}\,\cancel{|\vec r_u\times\vec r_v|}\,du\,dv$$

$\begin{pmatrix}\text{surface integral} \\ \text{of } \vec F \text{ over } S \\ \| \\ \text{flux of } \vec F \text{ across } S\end{pmatrix} = \boxed{\iint_D \left(\vec F\cdot\vec r_u\times\vec r_v\right)du\,dv}$

box product.

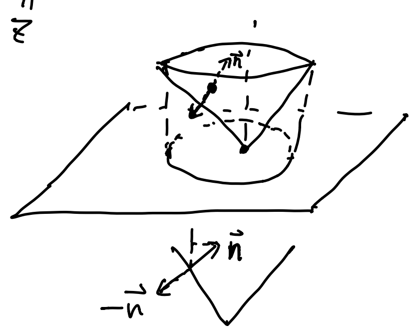


$$\iint_S \vec F\cdot(-\vec n)\,dA = -\iint_S \vec F\cdot\vec n\,dA$$

Ex: $\boxed{\vec F = x^2y\,\vec i + 2y^3z\,\vec j + z\cdot\vec k}$

surface S: $\vec r(r,\theta) = \underbrace{r\cos\theta}_{x}\,\vec i + \underbrace{r\sin\theta}_{y}\,\vec j + \underbrace{r}_{z}\,\vec k, \quad 0\le r\le 4,\ 0\le\theta\le 2\pi$

$x^2+y^2 = z^2 = r^2$



$$\begin{array}{cccc} & \vec i & \vec j & \vec k \\ \vec r_r = & \langle \cos\theta, & \sin\theta, & 1\rangle \\ \vec r_\theta = & \langle -r\sin\theta, & r\cos\theta, & 0\rangle \end{array}$$

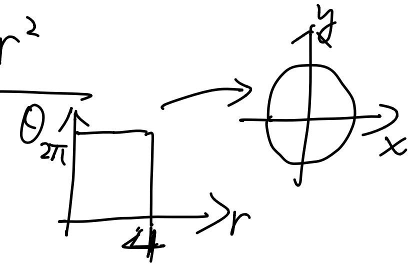
$\vec r_r\times\vec r_\theta = \langle -r\cos\theta, -r\sin\theta, \underbrace{r}_{r\cos^2\theta + r\sin^2\theta}\rangle$

$$\vec n = \frac{\vec r_r\times\vec r_\theta}{|\vec r_r\times\vec r_\theta|}$$

$$\vec{F}\cdot\vec{r_r}\times\vec{r_\theta} = x^2y\cdot(-r\cos\theta) + 2y^3z\cdot(-r\sin\theta) + z\cdot r$$

$$= (r\cos\theta)^2\cdot(r\sin\theta)(-r\cos\theta) + 2(r\sin\theta)^3\cdot r\cdot(-r\sin\theta) + r\cdot r$$

$$= -r^4\cdot\cos^3\theta\cdot\sin\theta - 2r^5\cdot\sin^4\theta + r^2$$



$$\iint_D (\vec{F}\cdot\vec{r_r}\times\vec{r_\theta})\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left(-r^4\cdot\cos^3\theta\cdot\sin\theta - 2r^5\sin^4\theta + r^2\right)dr\,d\theta$$

$$\left(\int_0^{2\pi}\cos^3\theta\,\sin\theta\,d\theta = -\int_0^{2\pi}\cos^3\theta\cdot d(\cos\theta) = -\frac{1}{4}\cos^4\theta\Big|_0^{2\pi} = 0\right)$$

$$= -2\int_0^{2\pi}\sin^4\theta\,d\theta\cdot\frac{1}{6}r^6\Big|_0^4 + 2\pi\cdot\frac{1}{3}\cdot r^3\Big|_0^4$$

$$= -\frac{1}{3}\cdot 4^6\cdot\int_0^{2\pi}\sin^4\theta\,d\theta + \frac{2\pi}{3}\cdot 4^3.$$

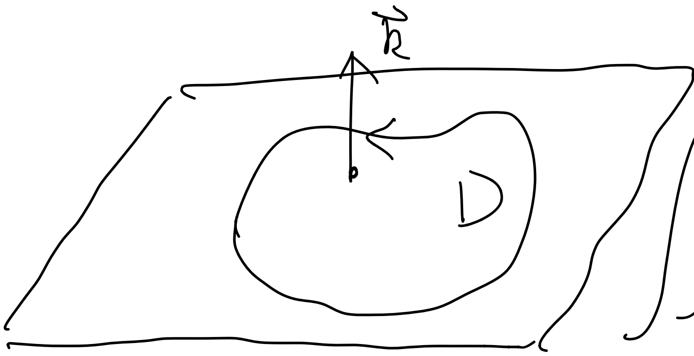
$\parallel$

$$\iint_\nabla \vec{F}\cdot\vec{n}\,dA$$

$$\iint_\nabla \nabla\times\vec{F}\cdot\vec{n}\,dA$$

Stokes' Theorem

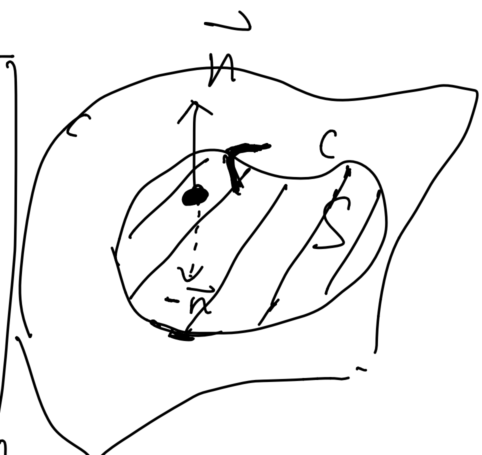
Green's Theorem: $\oint_C \vec{F} \cdot d\vec{r}$



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\,\vec{F} \cdot \vec{k}\, \underset{dxdy}{dA}$$

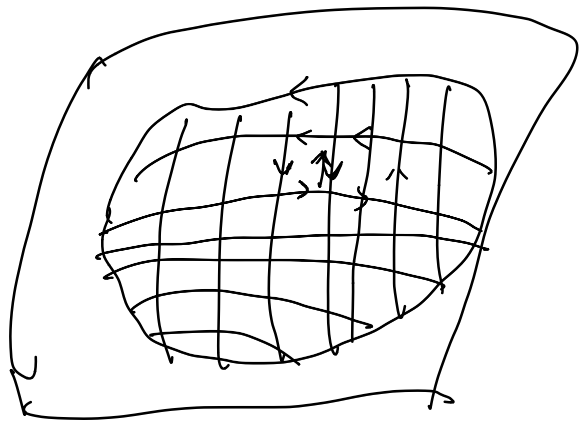
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dA$$

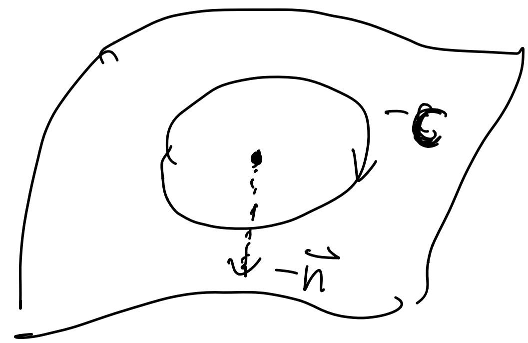
circulation density in direction $\vec{n}$

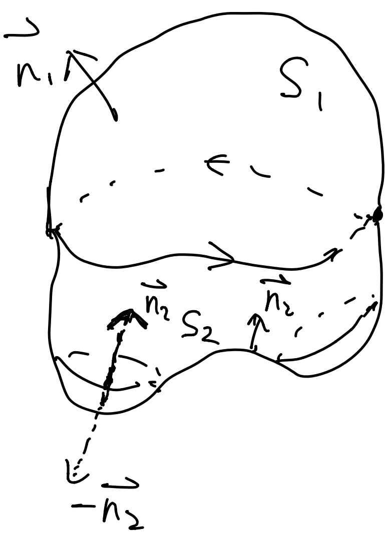


orientation of C is compatible with the orientation of S

right handed rule







$$\iint_{S_1} \text{curl}\,\vec F \cdot \vec n_1 \, dA$$

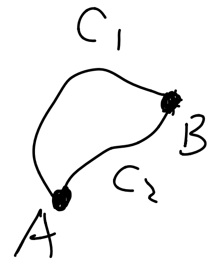
$$\parallel$$

$$\oint_C \vec F \cdot d\vec r$$

$$\parallel$$

$$\iint_{S_2} \text{curl}\,\vec F \cdot \vec n_2 \, dA$$

$$\vec F = \nabla f$$



$$\int_{C_1} \nabla f \cdot d\vec r$$

$$\parallel$$

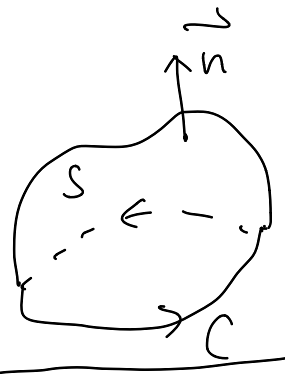
$$f(B) - f(A)$$

$$\parallel$$

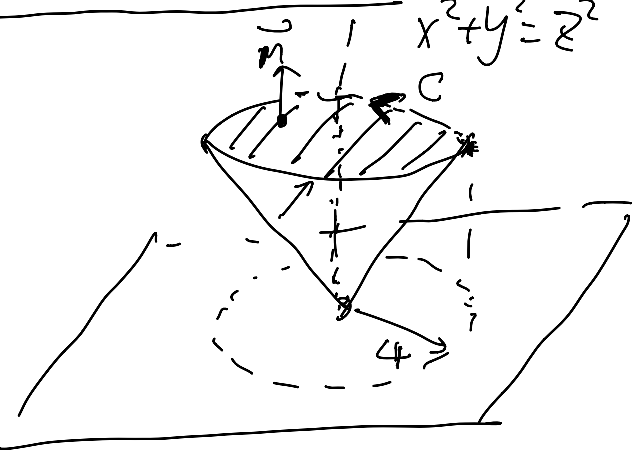
$$\int_{C_2} \nabla f \cdot d\vec r$$

$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{i}\cdot\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \vec{j}\cdot\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \vec{k}\cdot\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$



$$\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dA}$$



Ex:

$\vec{F} = x^2y\,\vec{i} + 2y^3z\,\vec{j} + z\cdot\vec{k}$

$C: \vec{r}(t) = \langle 4\cos\theta, 4\sin\theta, 4\rangle$

$\vec{r}\,'(t) = \langle -4\sin\theta, 4\cos\theta, 0\rangle$

$(r\cos\theta, \; r\sin\theta, r)$ $0\le r\le 4$, $0\le\theta\le 2\pi$

$$\vec{F}\cdot\vec{r}\,'(t) = x^2y\cdot(-4\sin\theta) + 2y^3z\cdot 4\cos\theta + z\cdot 0$$

$$= (4\cos\theta)^2\cdot 4\sin\theta\cdot(-4\sin\theta) + 2\cdot(4\sin\theta)^3\cdot 4\cdot 4\cos\theta$$

$$= -4^4\cdot\cos^2\theta\cdot\sin^2\theta + 2\times 4^5\cdot\sin^3\theta\cdot\cos\theta$$

$$\int \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi}\left(-4^4\cos^2\theta\cdot\sin^2\theta + \underline{2\times 4^5\sin^3\theta\cos\theta}\right)d\theta$$

$$\parallel$$

$$\int \vec{F}\cdot d\vec{r}$$

$$= -4^4\cdot\int_0^{2\pi}\frac{\sin(2\theta)^2}{4}\cdot d\theta$$

$$= -4^3\cdot\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= -4^{3}\times\frac{1}{2}\times 2\pi = -64\pi.$$

$$\nabla\times\vec{F} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & 2y^3z & z\end{vmatrix} = \vec{i}(-2y^3) - \vec{j}\cdot(0) + \vec{k}\cdot(-x^2)$$

$$\parallel$$

$$\underline{\mathrm{curl}\,\vec{F}}$$

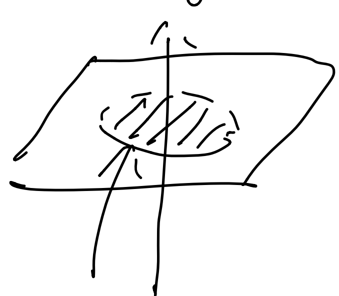
$$\parallel$$

$$\vec{k}\cdot\left(\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}x^2y\right)$$

$$\underline{\nabla\times\vec{F}\cdot\vec{k} = x^2}$$

$$\vec{k} = \vec{n}_1$$

$$\iint -x^2\,dA = -\int_0^{2\pi}\int_0^4 r^2\cos^2\theta\cdot r\,dr\,d\theta$$



$(r\cos\theta, r\sin\theta, 4)$

$0\le r\le 4,\ 0\le\theta\le 2\pi$

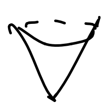
$$= -\left(\int_0^{2\pi} \cos^2\theta \, d\theta\right) \cdot \int_0^4 r^3 \, dr$$

$$= -\int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \, d\theta \cdot \frac{1}{4} \cdot r^4 \Big|_0^4$$

$$= -\frac{2\pi}{2} \cdot \frac{1}{4} \times 4^4 = -\pi \cdot 4^3 = \underline{-64\pi}.$$

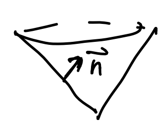
$$\iint_S \nabla \times \vec{F} \bullet \vec{n} \, \underline{dA} \quad \text{"} \; |\;| \, du\,dv \qquad \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\;|}$$

$$\iint_S = $$



$$= \underline{\iint \underline{(\nabla \times \vec{F})} \bullet \vec{r}_u \times \vec{r}_v \, du\,dv}$$

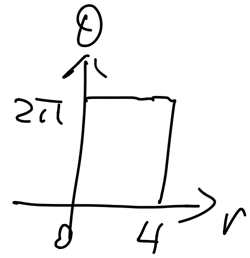
$$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r \rangle$$



$$\nabla \times \vec{F} = \vec{i}\,(-2y^3) + \vec{k} \cdot (-x^2)$$

$$\nabla \times \vec{F} \bullet \vec{r}_r \times \vec{r}_\theta = +2y^3 \cdot r\cos\theta + (-x^2) \cdot r \; \Big|_{(x,y,z) = (r\cos\theta,\, r\sin\theta,\, r)}$$

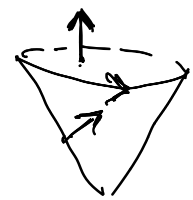
$$= 2\cdot r^3\sin^3\theta\cdot r\cos\theta - r^2\cos^2\theta\cdot r$$



$$-r^3\cos^2\theta$$

$$\int_0^{2\pi}\int_0^4\left(\underbrace{2\cdot r^3\sin^3\theta\cdot r\cos\theta}_{\parallel} + r\cancel{^2\cos^2\theta\cdot r\sin\theta}\right)dr\,d\theta$$

$?$ $\quad \underline{-64\pi}$ ✓



$$-\underbrace{\frac{1}{4}r^4\Big|_0^4}_{\parallel}\cdot\underbrace{\int_0^{2\pi}\cos^2\theta\,d\theta}_{\parallel}$$

$$-64\cdot\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}d\theta = \pi$$