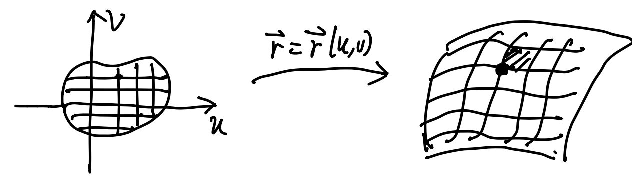
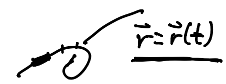
16.5 parametrization of surface

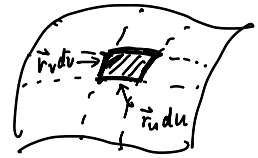




$$\int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt$$

ds: arclength

Area of surface



$$\lim_{|\square|\to 0} \sum \text{Area}(\square) = \iint_S \underset{\text{area diff.}}{dA}$$

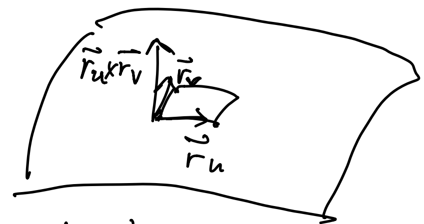
$$\| $$

$$\lim \sum |\vec{r}_u \times \vec{r}_v|\,du\,dv = \iint_D |\vec{r}_u\times\vec{r}_v|\,du\,dv$$

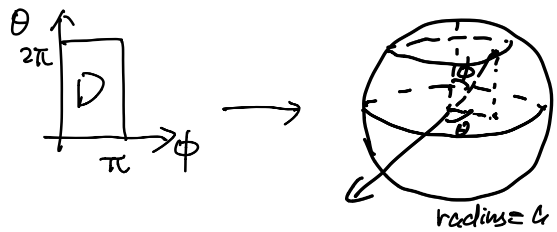
$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$, $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$

Area($\vec{r}_v dv$, $\vec{r}_u du$) $= |\vec{r}_u \times \vec{r}_v|\,du\,dv$

$$\boxed{\iint_S dA = \iint_D |\vec{r}_u\times\vec{r}_v|\,du\,dv}$$



Ex:



$\vec{r}(\phi,\theta) =$

$\langle a\cdot\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle$

$0 \le \phi \le \pi,\ 0\le\theta\le 2\pi$

$$\begin{array}{cccc} & \vec{i} & \vec{j} & \vec{k} \\ \vec{r}_\phi = \frac{\partial \vec{r}}{\partial \phi} = \langle & a\cos\phi\cos\theta, & a\cos\phi\sin\theta, & -a\sin\phi\rangle \\ \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \langle & -a\sin\phi\sin\theta, & a\sin\phi\cos\theta, & 0\rangle \end{array}$$

$$\vec{r}_\phi\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ - & - & - \\ - & - & - \end{vmatrix} = \vec{i}\cdot a^2\sin^2\phi\cos\theta + \vec{j}\cdot a^2\sin^2\phi\sin\theta + \vec{k}\cdot a^2\sin\phi\cos\phi \cdot \cancel{(\cos^2\theta+\sin^2\theta)}$$

$$|\vec{r}_\phi\times\vec{r}_\theta| = a^2\cdot\sqrt{\underbrace{\sin^4\phi\cos^2\theta+\sin^4\phi\sin^2\theta}_{\sin^4\phi}+\sin^2\phi\cos^2\phi} = a^2\sin\phi$$

$\sin^2\phi(\sin^2\phi+\cos^2\phi)$

$\leftrightarrow \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Area(S_a)

$\|$

$$\iint_D a^2\sin\phi\,d\phi\,d\theta = a^2\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = a^2\cdot 2\pi\cdot(-\cos\phi)\Big|_0^{\pi} = \boxed{4\pi a^2}$$

<u>Ex</u>: surface : graph of function $z=f(x,y)$

$\vec{r}(x,y)=\langle x,y,f(x,y)\rangle$

$\vec{i} \quad \vec{j} \quad \vec{k}$

$\vec{r}_x=\langle 1,0,f_x\rangle$

$\vec{r}_y=\langle 0,1,f_y\rangle$

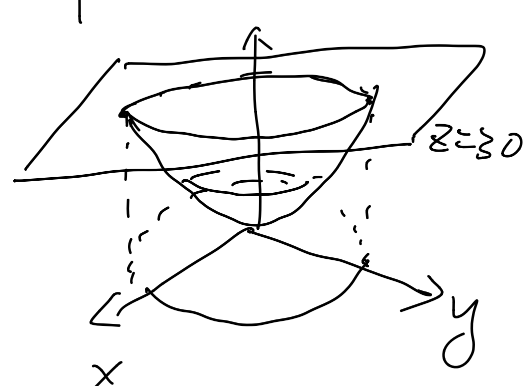
$\vec{r}_{\cancel{x}}\times\vec{r}_y=\langle -f_x,-f_y,1\rangle$

$|\vec{r}_x\times\vec{r}_y|=\sqrt{f_x^2+f_y^2+1}$

$$\boxed{\iint_S dA=\iint_D\sqrt{1+f_x^2+f_y^2}\,dx\,dy.}$$

<u>Ex</u>: The area of surface cut from the paraboloid $x^2+y^2-z=0$ by the plane $z=30$.

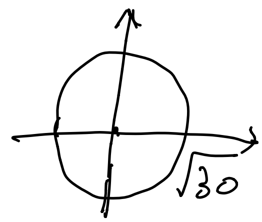
$z=x^2+y^2=f(x,y)$



$0\le z=\underbrace{x^2+y^2\le 30}_{\parallel\atop D}$

$$f_x = 2x,\quad f_y = 2y,\quad \sqrt{1+f_x^2+f_y^2} = \sqrt{1+4(x^2+y^2)}$$

$$\iint_D \sqrt{1+4(x^2+y^2)}\,\underline{dx\,dy}$$



$$\|$$

$$\int_0^{2\pi}\left(\int_0^{\sqrt{30}} \sqrt{1+4r^2}\,\underbrace{r\,dr}_{\frac{1}{2}dr^2}\right)d\theta = 2\pi\cdot\int_0^{30}\sqrt{1+4u}\cdot\frac{1}{2}\,du$$

$$\begin{array}{r} 121 \\ \times\ 11 \\ \hline 121 \\ 121 \\ \hline 1331 \end{array}$$

$$= \pi\cdot\int_0^{30}(1+4u)^{\frac{1}{2}}\,d\underline{(1+4u)}\cdot\frac{1}{4}$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}(1+4u)^{\frac{3}{2}}\Big|_0^{30} = \frac{\pi}{6}\left[121^{\frac{3}{2}}-1^{\frac{3}{2}}\right] = \frac{\pi}{6}(11^3-1) = \boxed{\frac{1330}{6}\pi}.$$

Ex: surface implicitly defined: $F(x,y,z)=0 \quad \xrightarrow{\frac{\partial F}{\partial z}\neq 0} \quad z = f(x,y)$

(Ex: $x^2+y^2-z^2=1$)

$\updownarrow$

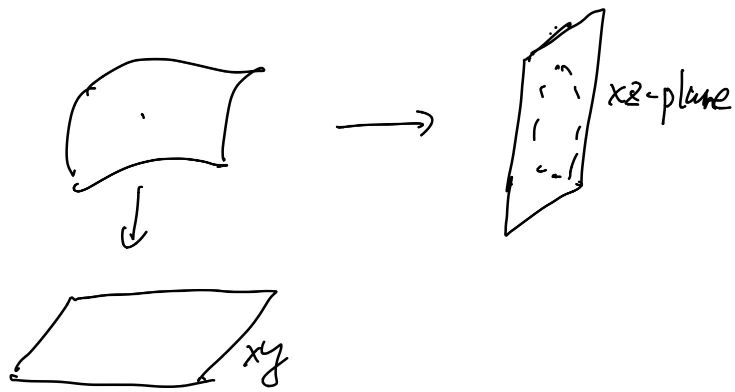
$z=\sqrt{x^2+y^2-1}$, or $y=\sqrt{1+z^2-x^2}$

$\frac{\partial F}{\partial z}\neq 0$, $\Rightarrow F_x + F_z\cdot f_x = 0 \Rightarrow f_x = -\frac{F_x}{F_z}$, Similarly $f_y = -\frac{F_x}{F_z}$

$F(x,y,f(x,y))=0$

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{F_x^2}{F_z^2}+\frac{F_y^2}{F_z^2}} = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|} = \frac{|\nabla F|}{|\nabla F\cdot\vec{k}|}$$

$$\boxed{\iint_S dA = \iint_D \frac{|\nabla F|}{|\nabla F\cdot\vec{k}|}\,dx\,dy}$$



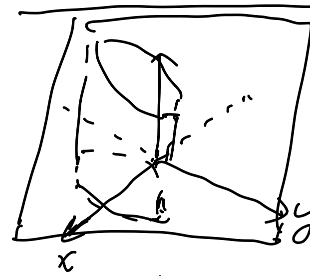
$$\iint_S dA = \iint_{D_{xz}} \frac{|\nabla F|}{|\nabla F \cdot \vec{j}|}\, dx\,dz = \iint_{D_{xz}} \sqrt{1+g_x^2+g_z^2}\, dx\,dz$$

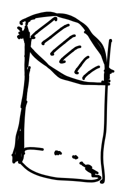
$y = g(x,z)$

$$\boxed{\iint_S dA = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\,dv}$$

Ex: Find the area of the region cut from the plane $2x+9y+6z=5$ by the cylinder whose walls are $x=y^2$ and $x=18-y^2$.

$$\iint_S dA = \iint_D \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|}\, dx\,dy$$

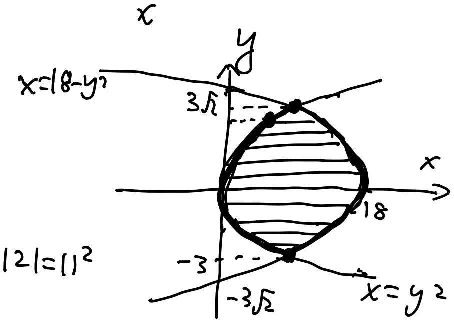




$F = 2x+9y+6z-5=0$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2, 9, 6 \rangle$

$F_z = \nabla F \cdot k$



$$= \iint_D \frac{\sqrt{4+81+36}}{6}\, dx\,dy$$

$85+36=121=11^2$

$$= \frac{11}{6}\int_{-3}^{3}\int_{y^2}^{18-y^2} dx\,dy = \frac{11}{3}\cdot\int_0^3 (18-2y^2)\,dy$$

$\begin{cases} x=y^2 \\ x=18-y^2 \end{cases} \Rightarrow 2y^2=18 \Rightarrow y=\pm 3$

$$= \frac{11}{3}\cdot\left(18y-\frac{2}{3}y^3\right)\Big|_0^3 = \frac{11}{3}\left(18\times 3-\frac{2}{3}\times 27\right) = \frac{11}{3}\cdot(54-18)$$

$$= \frac{11}{3}\times 36 = 11\times 12 = 132$$

16.6: surface integrals

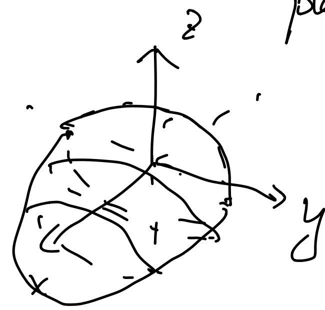
$$\iint_S f(x,y,z)\,dA = \iint_D f(x(u,v), y(u,v), z(u,v))\,\left|\vec{r}_u \times \vec{r}_v\right|\,du\,dv$$

EX: surface the portion of the paraboloid

$$g(y,z) = x = 10 - y^2 - z^2$$

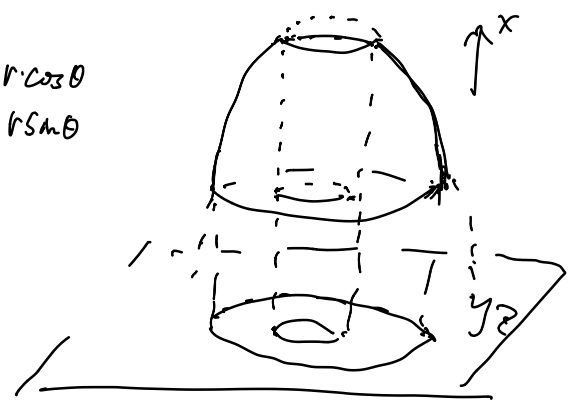
that lies above the ring $1 \le y^2 + z^2 \le 9$ in the yz plane

$f(x,y,z) = y$. $\quad g_y = -2y,\ g_z = -2z$



$$\iint_S f\,dA = \iint_D y \cdot \sqrt{1 + g_y^2 + g_z^2}\,dy\,dz$$

$$= \iint_D y \cdot \sqrt{1 + 4y^2 + 4z^2}\,dy\,dz$$



$y = r \cdot \cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi}\int_1^3 r\cdot\cos\theta \cdot \sqrt{1+4r^2}\cdot r\,dr\,d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{\sin\theta\big|_0^{2\pi}} \cdot \int_1^3 r\cdot\sqrt{1+4r^2}\,r\,dr$$

$= 0$

$1 \le y^2 + z^2 \le 9$