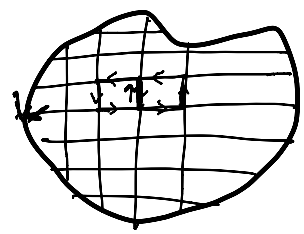
Green's theorem

$\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$. C : simple closed curve

- $\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl}\vec{F}\,dxdy}$ $\operatorname{curl}\vec{F} = \nabla\times\vec{F}\cdot\vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

$$\boxed{\oint M\,dx + N\,dy \qquad \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy}$$



circulation density

$$= \lim_{|\square|\to 0} \frac{\text{circulation along } \square}{\text{Area of } \square}$$

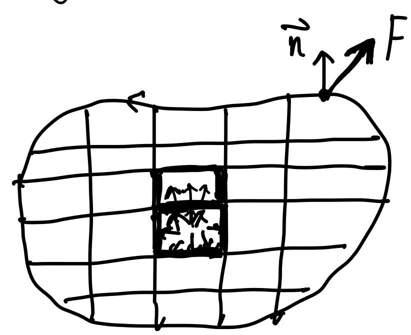
equivalent $\Updownarrow$

- $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D (\nabla\cdot\vec{F})\,dxdy$ $\nabla\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ divergence of $\vec{F}$

$$\oint_C M\,dy - N\,dx \qquad \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

| flux density

$$= \lim_{|\square|\to 0} \frac{\text{flux across } \square}{\text{Area of } \square}$$



$$\oint M\,dy - N\,dx = \oint (-N)\,dx + M\,dy$$

$$= \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial(-N)}{\partial y}\right) dxdy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy.$$

$\oint \vec{F}\cdot d\vec{r}$

$$= \oint_C M\,dx + N\,dy = \iint_D \boxed{d(M\,dx + N\,dy)} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

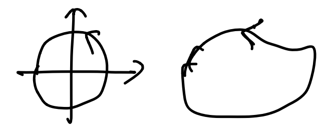
$$d(M\,dx + N\,dy) = dM\wedge dx + M\,\cancel{d(dx)} + dN\wedge dy + N\,\cancel{d(dy)}$$

$$= \left(\frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy\right)\wedge dx + \left(\frac{\partial N}{\partial x}dx + \frac{\partial N}{\partial y}dy\right)\wedge dy$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\boxed{d^2 = 0 \quad dx\wedge dx = 0 = dy\wedge dy = 0,\ dx\wedge dy = -dy\wedge dx}$

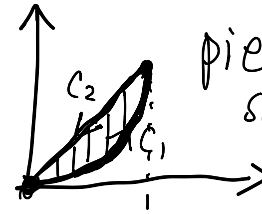
Ex: $\vec{F} = 5xy\,\vec{i} + 3y^2\,\vec{j}$



counterclockwise

D region bounded by curves $y = x^2$ and $y = x$

Circulation: $\oint_C \vec{F} \cdot d\vec{r} = \left(\int_{C_1} + \int_{C_2} \right) \vec{F} \cdot d\vec{r}$



piecewise smooth curve

$$\| $$

$$\iint_D (\text{curl } \vec{F})\, dx\, dy$$

$$\|$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(3y^2) - \frac{\partial}{\partial y}(5xy) = -5x.$$

$$\iint_D (-5x)\, dx\, dy = \int_0^1 \int_{x^2}^{x} (-5x)\, dy\, dx$$

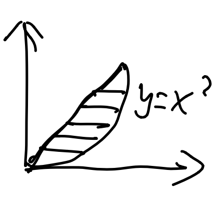
$$= \int_0^1 -5x \cdot y \Big|_{x^2}^{x} dx = \int_0^1 (-5x^2 + 5x^3)\, dx$$

$$= \left(-\frac{5}{3}x^3 + \frac{5}{4}x^4\right)\Big|_0^1 = -\frac{5}{3} + \frac{5}{4} = -\frac{5}{12}$$

Flux: $\oint \vec{F} \cdot \vec{n}\, ds = \iint_D (\nabla \cdot \vec{F})\, dx\, dy$

$\uparrow$ arclength

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x}(5xy) + \frac{\partial}{\partial y}(3y^2) = 5y + 6y = 11y$$



$$\iint_D (11y)\, dx\, dy = \int_0^1 \int_y^{\sqrt{y}} 11y\, dx\, dy$$

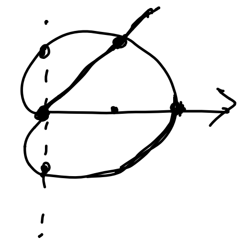
$$= 11 \int_0^1 yx \Big|_y^{\sqrt{y}} dy = 11 \cdot \int_0^1 \left(y^{\frac{3}{2}} - y^2\right) dy$$

$$= 11 \cdot \left(\frac{2}{5} y^{\frac{5}{2}} - \frac{1}{3} y^3\right)\Big|_0^1 = 11 \cdot \left(\frac{2}{5} - \frac{1}{3}\right) = 11 \times \frac{1}{15} = \frac{11}{15}.$$

Ex: $\vec{F} = \left(5xy - \frac{3x}{1+y^2}\right)\vec{i} + \left(e^x + 3\cdot\tan^{-1}y\right)\vec{j}$

curve: Cardioid $r = 1+\cos\theta$.

Flux: $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D (\nabla\cdot\vec{F})\,dx\,dy$



$$\nabla\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x}\left(5xy - \frac{3x}{1+y^2}\right) + \frac{\partial}{\partial y}\left(e^x + 3\cdot\tan^{-1}y\right)$$

$$= 5y - \cancel{\frac{3}{1+y^2}} + 3\cdot\cancel{\frac{1}{1+y^2}} = 5y$$

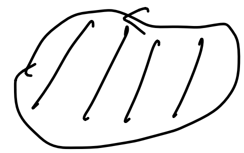
$$\iint_D (5y)\,dx\,dy = \int_0^{2\pi}\int_0^{1+\cos\theta} (5\cdot r\cdot\sin\theta)\,r\,dr\,d\theta$$

$$= 5\int_0^{2\pi} \sin\theta\cdot\frac{r^3}{3}\Big|_0^{1+\cos\theta} d\theta \qquad \begin{matrix}\sin\theta\,d\theta \\ \parallel \\ -d\cos\theta\end{matrix}$$

$$= \frac{5}{3}\cdot\int_0^{2\pi} (1+\cos\theta)^3 (-d\cos\theta)$$

$$= -\frac{5}{3}\int_1^1 (1+u)^3\,du = 0 \qquad -\frac{d}{d\theta}\left(\frac{1}{4}(1+\cos\theta)^4\right)$$

Green's Thm Area Formula.

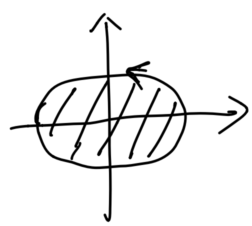


$$\frac{1}{2}\oint_C x\,dy - y\,dx \overset{\text{Normal form}}{=} \frac{1}{2}\iint_D \underbrace{\left(\frac{\partial}{\partial x}x + \frac{\partial y}{\partial y}\right)}_{=\,2}dxdy$$

$$x\,dy - y\,dx = (-y)\,dx + x\,dy$$

$$\boxed{\frac{1}{2}\oint_C x\,dy - y\,dx = \iint_D dxdy}$$

Ex:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \begin{cases} x = a\cdot\cos\theta \\ y = b\cdot\sin\theta \end{cases}$$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cdot\cos\theta)\cdot d(b\sin\theta) - (b\cdot\sin\theta)\cdot d(a\cos\theta)$$

$$= \frac{1}{2}\cdot\int_0^{2\pi} (ab)\cdot\cos\theta\cdot\cos\theta\,d\theta - (ba)\sin\theta(-\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\cdot\big(\underbrace{\cos^2\theta + \sin^2\theta}_{=\,1}\big)\,d\theta$$

$$= \frac{ab}{2}\cdot\int_0^{2\pi} d\theta = \frac{ab}{2}\,2\pi = \boxed{\pi ab} = \text{Area of ellipse}$$

Norm form: $$\oint M\,dy - N\,dx = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

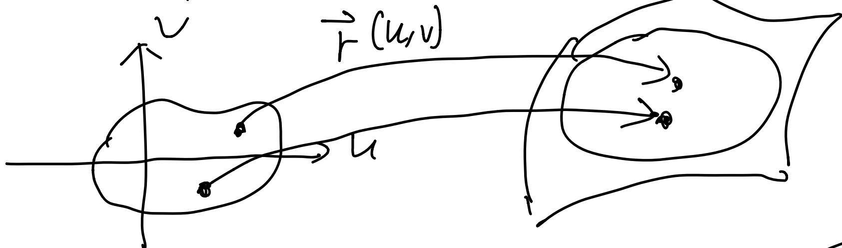
$$\oint \vec{F}\cdot\vec{n}\,ds \qquad = \qquad \iint_D (\nabla\cdot\vec{F})\,dx\,dy$$

Tangential form: $$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

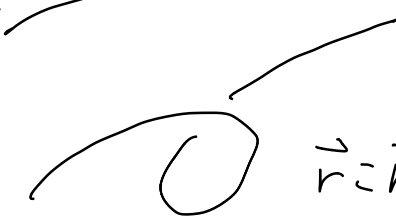
$$\oint \vec{F}\cdot d\vec{r} \qquad = \qquad \iint_D (\nabla\times\vec{F}\cdot\vec{k})\,dx\,dy$$

$$= \oint \vec{F}\cdot\vec{T}\,ds$$

16.5 Surfaces and Area



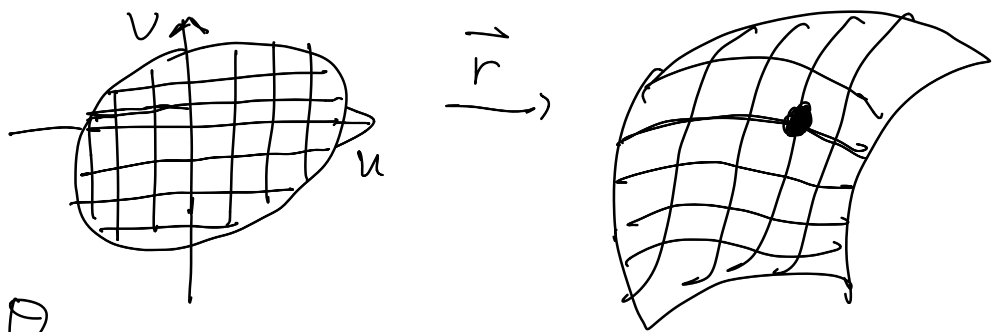
(length $S(t) = \int_a^t |\vec{r}\,'(t)|\,dt$

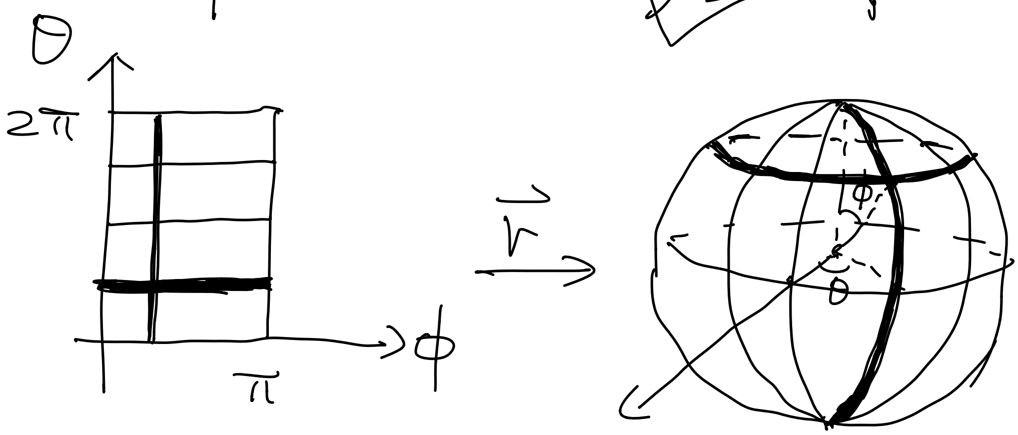


$\vec{r} = \vec{r}(t)$)

$$\vec{r}: \underset{\substack{\cap \\ uv\text{-plane}}}{D} \longrightarrow \mathbb{R}^3$$

$$(x(u,v),\ y(u,v),\ z(u,v))$$





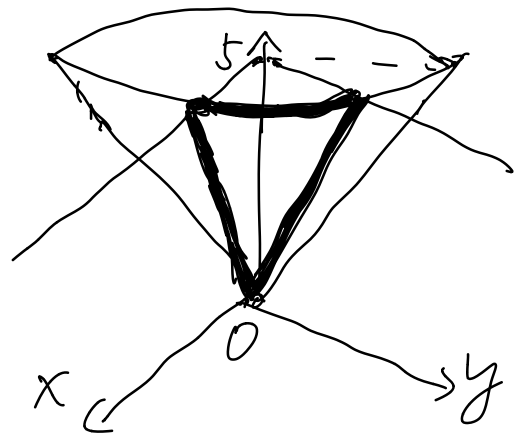
$$\vec{r}(\theta,\phi) = \langle a\cdot\sin\phi\cos\theta,\ a\cdot\sin\phi\sin\theta,\ a\cdot\cos\phi\rangle \qquad \begin{array}{l} 0 \le \phi < \pi \\ 0 \le \theta < 2\pi \end{array}$$

Ex: parametrize the surface: the 1st-Octant portion of the cone $z = \frac{\sqrt{x^2+y^2}}{6}$ between the planes $z=0$ and $z=5$

$x = r\cdot\cos\theta$

$y = r\cdot\sin\theta$

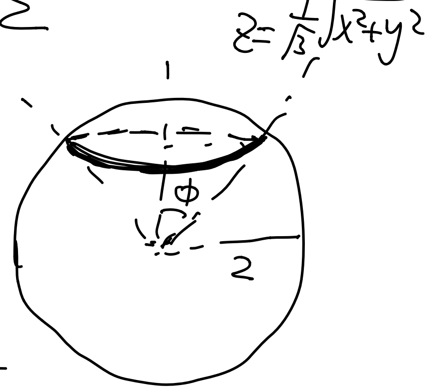
$z = \dfrac{r}{6}$



$0 \le z = \dfrac{r}{6} \le 5 \iff \underline{0 \le r \le 30}$

$0 \le \theta \le \dfrac{\pi}{2}$

<u>Ex</u>: Spherical Cap



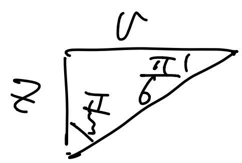
$x = 2\cdot\sin\phi\cos\theta$

$y = 2\cdot\sin\phi\sin\theta$

$z = 2\cdot\cos\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \dfrac{\pi}{3}$



$\tan^{-1}(\sqrt{3}) \Leftarrow \dfrac{r}{|z|} = \sqrt{3}$