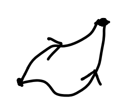
$\vec{F} = \vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$

$\vec{F}$ is conservative on D if $\int_C \vec{F}\cdot d\vec{r}$ is path independent



<u>Thm:</u> $\vec{F}$ is <u>conservative</u> $\Leftrightarrow$ $\vec{F} = \nabla f$ for some potential function f

"$\Rightarrow$": $f(P) = \int_A^P \vec{F}\cdot d\vec{r}$

"$\Leftarrow$" $\underline{\int_C \vec{F}\cdot d\vec{r}} = \int_C \nabla f\cdot d\vec{r} = \int_C \frac{d}{dt} f(\vec{r}(t))\,dt = \underline{f(B) - f(A)}$.

Component test for $\vec{F}$ being conservative:

- $\vec{F}$ conservative $\Rightarrow$ $\nabla\times\vec{F} = \vec{0} \Leftrightarrow \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x},\ \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}.}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{i}\cdot\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \vec{j}\cdot\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \vec{k}\cdot\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

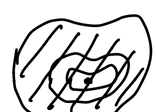
Thm: $\boxed{\nabla\times\vec{F} = \vec{0}}$ and D is simply connected $\Rightarrow$ $\vec{F}$ is conservative

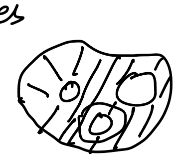
D: domain

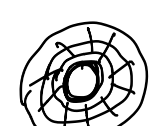
simply connected $\Leftrightarrow$ no holes

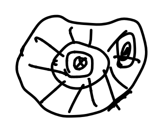
Topology:

$\boxed{\text{simply connected:}}$ <u>every closed curve</u> can be continuously shrinked to a point.







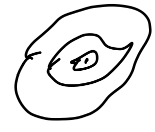


not simply connected

<u>Ex:</u> $\underline{\vec{F} = (y+3z)\vec{i} + (x+5z)\vec{j} + (3x+5y)\vec{k}}$. defined every where

D = plane simply connected

<u>Test:</u> $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+3z & x+5z & 3x+5y \end{vmatrix}$



$= \vec{i}\cdot(\underset{\frac{\partial}{\partial y}(3x+5y)}{5} - \underset{\frac{\partial}{\partial z}(x+5z)}{5}) + \vec{j}\,(3-3) + \vec{k}\cdot(1 - \underset{\frac{\partial}{\partial z}(y+3z)}{1}) = \vec{0}$. ✓

$\Rightarrow$ $\vec{F}$ is <u>conservative</u>

Find a potential function: f s.t. $\vec{F} = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

system $\rightarrow$
$$\begin{cases} \boxed{\frac{\partial f}{\partial x} = y + 3z} \Rightarrow f = \int (y+3z)\,dx = (y+3z)x + \boxed{g(y,z)} \quad \text{const. w.r.t. } x. \\ \boxed{\frac{\partial f}{\partial y} = x + 5z} \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + 5z \Rightarrow \frac{\partial g}{\partial y} = 5z \\ \frac{\partial f}{\partial z} = 3x + 5y \end{cases}$$

$$\Downarrow$$

$$g(y,z) = \int 5z\,dy$$

$$\boxed{g(y,z) = 5zy + h(z)}$$

$$f(x,y,z) = (y+3z)x + g(y,z)$$

$$= \underline{(y+3z)x + 5zy} + h(z)$$

$$\frac{\partial f}{\partial z} = 3x + 5y + \frac{dh}{dz} = 3x + 5y \Rightarrow \frac{dh}{dz} = 0 \Rightarrow \underline{h = C}$$

choose $\underline{C = 0}$.

$$\Rightarrow f(x,y,z) = (y+3z)x + 5zy = xy + 3xz + 5yz$$

Ex: $\vec{F} = e^{y+2z}(3\vec{i} + 3x\vec{j} + 6x\vec{k})$

$$\begin{cases} \frac{\partial f}{\partial x} = e^{y+2z} \cdot 3 \Rightarrow f = 3\int e^{y+2z}\,dx = 3x \cdot e^{y+2z} + \boxed{g(y,z)} \\ \frac{\partial f}{\partial y} = e^{y+2z} \cdot 3x \qquad \frac{\partial f}{\partial y} = 3x \cdot e^{y+2z} + \frac{\partial g}{\partial y} = e^{y+2z} \cdot 3x \\ \frac{\partial f}{\partial z} = e^{y+2z} \cdot 6x \end{cases}$$

$$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$$

$$f = 3x e^{y+2z} + h(z) \qquad \frac{\partial f}{\partial z} = 3x \cdot e^{y+2z} \cdot 2 + h'(z)$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C. \qquad \underline{6x e^{y+2z}}$$

Choose $C = 0$. $\Rightarrow \boxed{f(x,y,z) = 3xe^{y+2z}} \quad \nabla f = \vec{F}$

Ex: $\vec{F} = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$, $(x,y) \neq (0,0)$.

conservative or not? $\quad \frac{\partial N}{\partial x} \overset{?}{=} \frac{\partial M}{\partial y}$

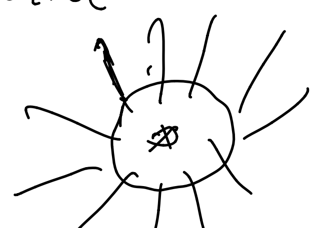
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \vec{i}\cdot 0 + \vec{j}\cdot 0 + \boxed{\vec{k}\cdot\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$$

$$\frac{\partial N}{\partial x} = \underbrace{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)}_{x \leftrightarrow y} = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \ominus\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad //$$

$\nabla \times \vec{F} = \vec{0} \;\overset{?}{\not\Rightarrow}\; \vec{F}$ is conservative

D is simply connected?



Claim: $\vec{F}$ is not conservative on $\mathbb{R}^2 \setminus 0$

punctured plane.

not simply connected

$\Updownarrow$

$\int_C \vec{F}\cdot d\vec{r}$ is path dependent.

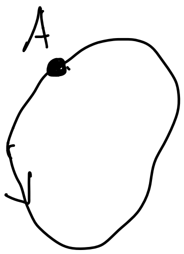
$\Updownarrow$

there exists closed curve

$\exists$ curve $\oint_C \vec{F}\cdot d\vec{r} \neq 0$

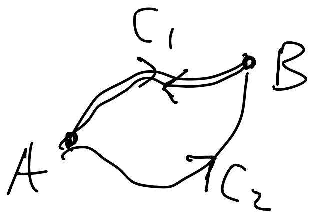
$\boxed{\text{Fact!}}$ $\vec{F}$ is conservative on D $\Longleftrightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ path independ.

$$\Updownarrow \qquad\qquad \Updownarrow$$



$\oint_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve in D.

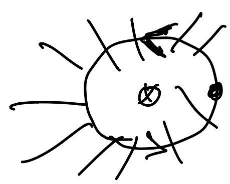
$$\oint \nabla f\cdot d\vec{r} = f(A) - f(A) = 0$$



$$\int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r}$$

$$\int_{\substack{C \\ \| \\ C_2 \cup (-C_1)}} \vec{F}\cdot d\vec{r} = 0$$

$$\vec{F} = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$



$\vec{r}(t) = \langle \cos t, \sin t\rangle$

$0 \le t < 2\pi$

$$\oint \vec{F}\cdot d\vec{r} = \oint M\,dx + N\,dy$$

$$= \int_0^{2\pi} -\frac{y}{x^2+y^2}\Big|_{(x,y)=(\cos t,\sin t)} \cdot d\cos t + \frac{x}{x^2+y^2}\Big|_{(x,y)=(\cos t,\sin t)} d\cdot\sin t.$$

$$= \int_0^{2\pi} -\frac{\sin t}{\underbrace{\cos^2 t+\sin^2 t}_{=1}}\cdot(-\sin t)\,dt + \frac{\cos t}{\underbrace{\cos^2 t+\sin^2 t}_{=1}}\cos t\cdot dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi \neq 0.$$

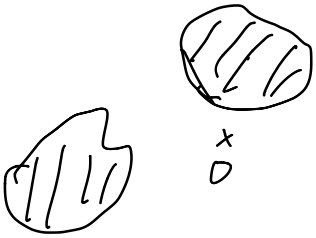
$$\Rightarrow \vec{F} = -\frac{y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$$ is not cons. on $\mathbb{R}^2 \setminus \{0\}$

not simply connected

even though $\underline{\nabla \times \vec{F} = \vec{0}}$.

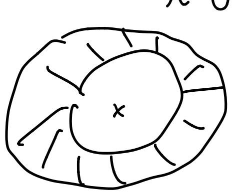
$$\begin{cases} \dfrac{\partial f}{\partial x} = -\dfrac{y}{x^2+y^2} \Rightarrow f(x,y) = \displaystyle\int -\frac{y}{x^2+y^2}\,dx = -\tan^{-1}\left(\frac{x}{y}\right) + g(y) \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{\partial f}{\partial y} = \dfrac{x}{x^2+y^2} + g'(y) \Rightarrow g' = 0 \Rightarrow g = C. \end{cases}$$

$$f(x,y) = -\tan^{-1}\left(\frac{x}{y}\right) = -\mathrm{Arctan}\left(\frac{x}{y}\right).$$ not continuous defined globally on $\mathbb{R}^2 \setminus \{0\}$



$\vec{F}$ is conservative on any $\underline{\text{simply connected}}$ region that does not contain the origin. by the theorem.

$$\vec{F} = -\frac{y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$$



16.4 Green's theorem

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

integration over the region enclosed by the bdry.

integration over the bdry.

$$\iint_D \underbrace{(\nabla\times\vec{F}\cdot\vec{k})}_{=\ \mathrm{curl}\,\vec{F}}\,dx\,dy = \oint_C \vec{F}\cdot d\vec{r}$$

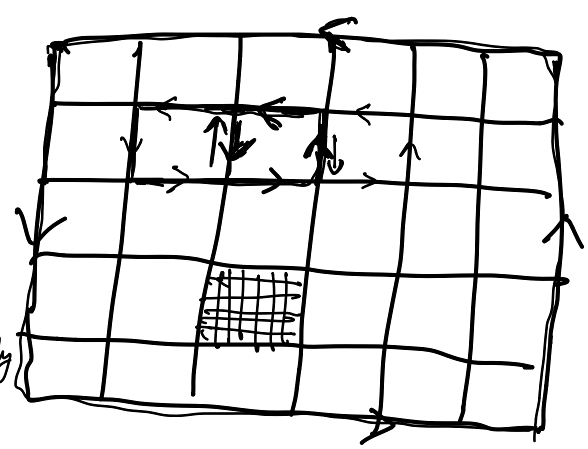
$$\vec{F} = M\vec{i} + N\vec{j}$$

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \overset{\text{Green's thm}}{=} \oint_C M\,dx + N\,dy$$

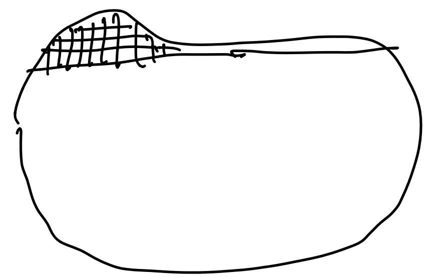
$$\nabla\times\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$\oint \vec{F}\cdot d\vec{r} = \lim \sum_{\square} \oint_{\square} \vec{F}\cdot d\vec{r}$$

$$= \iint_D \underbrace{(\text{circulation density})}_{=\ \mathrm{curl}\,\vec{F}}\,dx\,dy$$



$$\text{curl } \vec{F} = \nabla \times \vec{F} \cdot \vec{k}$$
$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$



<u>15.8-16.5</u> : Mod 4.