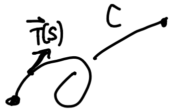
$$\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k} = \underline{\langle M,\ N,\ P\rangle}$$



$\vec{r} = \vec{r}(t)$, $a \le t \le b$

$= \vec{r}(s)$, $s_{min} \le s \le s_{max}$

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \left(\vec{F} \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$\parallel$$

$$\int_C \vec{F} \bullet \frac{d\vec{r}}{ds}\, ds$$

$$\parallel$$

$$\int_C \vec{F} \bullet \vec{T}\, ds$$

flow of $\vec{F}$ along C

$\parallel$

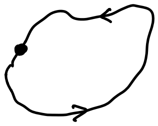
Work done by the force along C.

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

$$d\vec{r}(t) = \underset{\substack{\parallel \\ x'(t)dt}}{(dx)}\,\vec{i} + \underset{\substack{\parallel \\ y'(t)dt}}{(dy)}\,\vec{j} + \underset{\substack{\parallel \\ z'(t)dt}}{(dz)}\,\vec{k}$$

$$\vec{F} \bullet d\vec{r} = M\,dx + N\,dy + P\,dz$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \overset{\text{differential form}}{M\,dx + N\,dy + P\,dz}$$

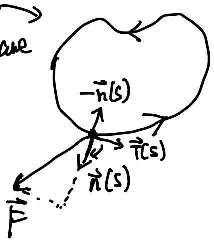


$$\int_C \vec{F} \bullet d\vec{r} : \text{ circulation along a closed curve.}$$

$$\parallel$$

$$\int_C \vec{F} \bullet \vec{T}\, ds = \int_C M\,dx + N\,dy$$

Simple closed curve on the plane



$$\int_C \vec{F} \bullet \vec{n}\, ds : \text{ flux of } \vec{F} \text{ across } C$$

$$\parallel$$

$$\int_C \langle M, N\rangle \bullet \left\langle \frac{dy}{ds}, -\frac{dx}{ds}\right\rangle ds = \int_C \boxed{M\,dy - N\,dx.}$$

$$/\!/ \qquad \vec{r}(t),\ a \le t \le b.$$

$$\underset{\text{flux.}}{\int_C \vec{F} \bullet \vec{n}\, ds} = \underline{\int_a^b \left[M(\vec{r}(t))\frac{dy}{dt} - N(\vec{r}(t))\frac{dx}{dt}\right] dt.}$$

$$\vec{T}(s) = \frac{d\vec{r}}{ds} = \langle x'(s), y'(s)\rangle$$

$$\vec{n}(s) = \cancel{\langle -y'(s), x'(s)\rangle}$$

$$\boxed{\langle y'(s), -x'(s)\rangle}$$

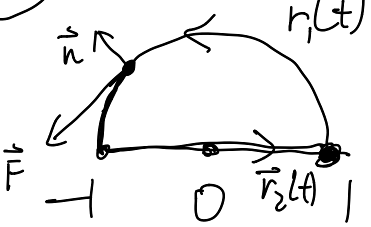
$$\parallel$$

$$\vec{T} \times \vec{k}$$

$$\parallel$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle y', -x', 0\rangle$$

$\underline{\text{Ex}}$: $\vec{F} = \underline{-3y}\,\vec{i} + \boxed{3x}\,\vec{j}$



$\underline{\vec{r}_1(t) = \cos t\cdot\vec{i} + \sin t\cdot\vec{j}}$, $0\le t\le\pi$

$\vec{r}_2(t) = t\,\vec{i}$, $-1\le t\le 1$

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C M\,dy - N\,dx$$

$$= \int_{C_1}(M\,dy - N\,dx) + \int_{C_2}(M\,dy - N\,dx)$$

$\underline{\int_{C_1}\vec{F}\cdot\vec{n}\,ds}$

$\parallel$

$$\int_{C_1}\boxed{M\,dy - N\,dx} = \int_{C_1}\underset{M}{\underset{\parallel}{(-3y)}}\Big|_{y=\sin t}\,d(\sin t) - \underset{N}{\underset{\parallel}{3x}}\Big|_{x=\cos t}\cdot d(\cos t)$$

$$= \int_0^\pi \underline{-3\cdot\sin t\cdot\cos t}\cdot dt - 3\cdot\cos t\cdot(-\sin t)\,dt$$

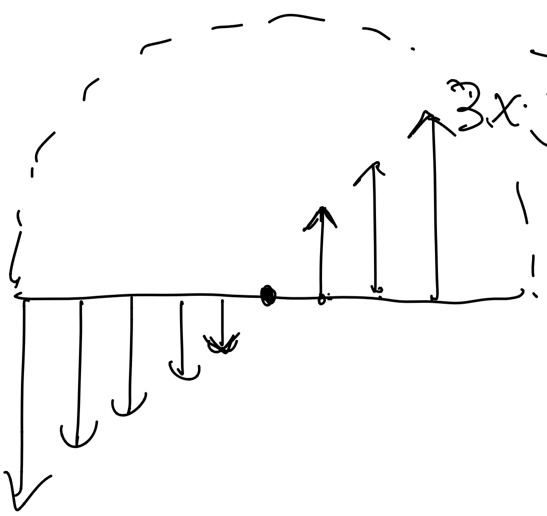
$$= \int_0^\pi 0\,dt = 0. \qquad \left(\begin{aligned}\vec{F}\cdot\vec{n} &= \langle -3y, 3x\rangle\cdot\langle x, y\rangle\\ &= -3yx + 3xy = 0\end{aligned}\right)$$

$$\int_{C_2} M\,dy - N\,dx = \int_{C_1}(-3y)\big|_{y=0}\cdot d0 - 3x\big|_{x=t}\cdot dt$$

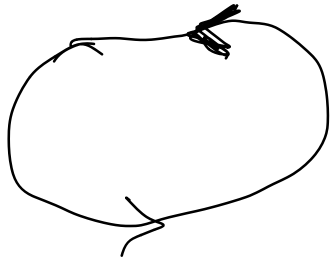
$\vec{r}_2(t) = t\vec{i}$

$-1\le t\le 1$

$$= \int_{-1}^1 -3t\,dt = -\frac{3}{2}t^2\Big|_{-1}^1 = \underline{0}.$$



$$3x\vec{j} = \vec{F} = -3y\vec{i}\big|_{y=0} + 3x\vec{j}$$



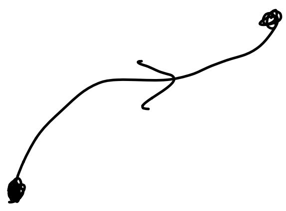
use counter-clockwise

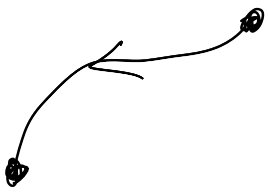
$$\oint \vec{F}\cdot\vec{n}\,ds$$

$$\parallel$$

$$\oint_C M\,dy - N\,dx$$

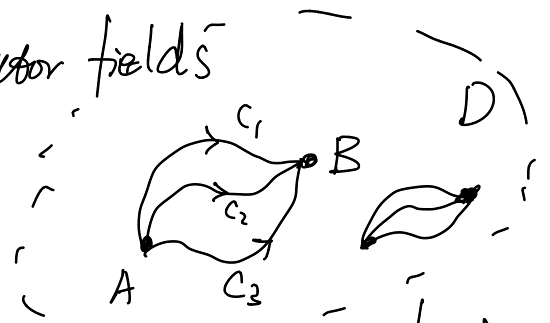
$$-\int_C \vec{F}\cdot d\vec{r} = \int_{-C} \vec{F}\cdot d\vec{r}$$





Conservative Vector fields

$\vec{F} = \vec{F}(x,y,z)$

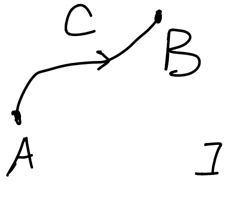


Def: $\int_C \vec{F} \cdot d\vec{r}$ is path independent if the integral only depends on the starting and ending pts of C, but does not depend on the path from starting pt. to the end point.

$\vec{F}$ is called conservative on D if $\int_C \vec{F} \cdot d\vec{r}$ is path independent for any two points in D.

Def: If $\vec{F} = \nabla f$ for some scalar function f on D then f is called a potential function for $\vec{F}$.

Theorem (Fundamental Thm of line integrals)



$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

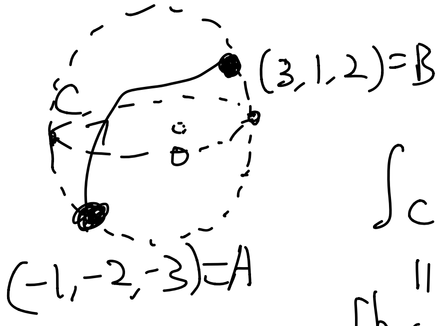
In particular, $\boxed{\nabla f \text{ is conservative}}$

Proof: $\vec{r} = \vec{r}(t)$, $a \le t \le b$, $\vec{r}(a) = A$, $\vec{r}(b) = B$

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left(\nabla f \cdot \frac{d\vec{r}}{dt}\right) dt = \int_a^b \left(\frac{d}{dt} f(\vec{r}(t))\right) dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A)$$

Ex: $\underline{f = \frac{1}{2}\ln(x^2+y^2+z^2)}$, $\nabla f = \underline{\langle \underset{\overset{\parallel}{\frac{\partial f}{\partial x}}}{\frac{x}{x^2+y^2+z^2}}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\rangle}$



$$\int_C \nabla f \cdot d\vec{r} = \underline{f(B)-f(A)} = \frac{1}{2}\ln(3^2+1^2+2^2) - \frac{1}{2}\ln((-1)^2+(-2)^2+(-3)^2)$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$\int_a^b (\nabla f \cdot \vec{r}\,'(t))\,dt \qquad\qquad\qquad 0$$

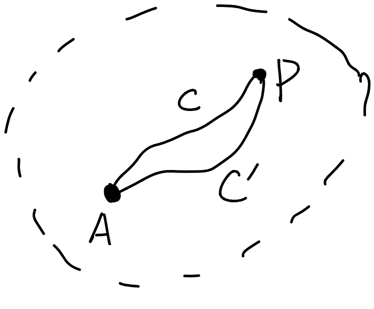
$$\parallel$$

$$\left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)$$

Thm: Conservative vector fields $\underset{\text{are}}{\Longleftrightarrow}$ gradient vector fields

Pf: Suppose $\vec{F}$ conservative.

We want to construct a $\underline{\text{function } f}$ such that $\vec{F} = \nabla f$.



$f(P) = \int_C \vec{F}\cdot d\vec{r}$ $\underline{\text{is well-defined}}$

$= \int_{C'} \vec{F}\cdot d\vec{r}$ $\forall P \in D$

$f(A) = 0$.

Finally verify that $\underline{\nabla f = \vec{F}}$. ▨

Q1: How do we test whether $\vec{F}$ is conservative?

Q2: How do we calculate a potential function of $\vec{F}$ if $\vec{F}$ is conservative?

$$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k} = \nabla f$$

$$\begin{array}{c} \parallel \\ f_x\vec{i} + f_y\vec{j} + f_z\vec{k} \end{array}$$

$$\Leftrightarrow \quad f_x = M, \quad f_y = N, \quad f_z = P$$

$$\Rightarrow \quad \begin{array}{ccccc} f_{xy} & = & \frac{\partial f_x}{\partial y} & = & \frac{\partial M}{\partial y} \\ \parallel & & & & \parallel \\ f_{yx} & = & \frac{\partial f_y}{\partial x} & = & \frac{\partial N}{\partial x} \end{array} \qquad \begin{array}{c} \frac{\partial M}{\partial z} \\ \parallel \\ \frac{\partial P}{\partial x} \end{array} = f_{xz} \qquad \begin{array}{c} \frac{\partial N}{\partial z} \\ \parallel \\ \frac{\partial P}{\partial y} \end{array} = f_{yz}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \boxed{\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)}\vec{i} + \boxed{\left(-\frac{\partial P}{\partial x} + \frac{\partial M}{\partial z}\right)}\vec{j} + \boxed{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}\vec{k}$$

$$\begin{array}{c} \parallel \\ \nabla \times \vec{F} \\ \text{(curl of } \vec{F}\text{)}. \end{array}$$

$\vec{F}$ is conservative $\Longrightarrow \boxed{\nabla \times \vec{F} = \vec{0}}$.

$\overset{?}{\Longleftarrow}$

$$\nabla\times\langle z,x,y\rangle=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ z&x&y\end{vmatrix}=\vec{i}\cdot\left(\overset{1}{\overset{\shortparallel}{\frac{\partial y}{\partial y}}}-\overset{0}{\overset{\shortparallel}{\frac{\partial x}{\partial z}}}\right)+\cdots$$

$$=\boxed{\vec{i}\cdot 1}+\cdots\neq\vec{0}$$

$\Rightarrow \langle z,x,y\rangle$ is not conservatie.

$$\vec{F}=M(x,y)\,\vec{i}+\underline{N(x,y)}\,\vec{j}\,.$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ M&N&0\end{vmatrix}=0\cdot\vec{i}+0\cdot\vec{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\vec{k}\,.$$

$$=\boxed{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}\,\vec{k}$$

Thm: Assume D is simply connected.

$\nabla\times\vec{F}=\vec{0} \Longrightarrow \vec{F}$ is conservative.

Ex: $\vec{F} = y\vec{i} + x\vec{j}$

$$\boxed{\nabla \times \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k} = 0.}$$

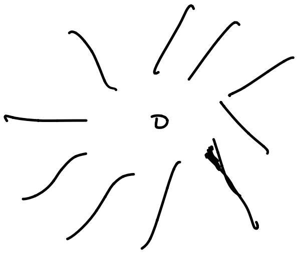
$$\frac{\partial N}{\partial x} = \frac{\partial x}{\partial x} = 1$$

//

$$\frac{\partial M}{\partial y} = \frac{\partial y}{\partial y} = 1$$

$$\vec{F} = \nabla(x\cdot y) = \frac{\partial(xy)}{\partial x}\vec{i} + \frac{\partial(xy)}{\partial y}\vec{j} = y\vec{i} + x\cdot\vec{j}$$

Ex: $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2} = -\dfrac{y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j}$



not simply connected