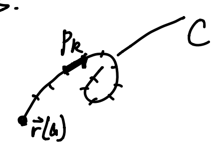
16.1: line integrals of scalar functions.



$f = f(x,y,z)$ $\qquad$ $\underline{\vec{r} = \vec{r}(t)}$; $a \le t \le b$.

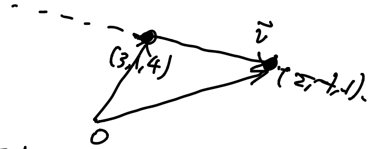
$$\int_C f\, ds = \lim_{\|\Delta s\| \to 0} \sum_k f(P_k)\Delta s_k$$

↑ arclength parameter

$$s = s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau \qquad \underline{ds} = \frac{ds}{dt}\, dt = |\vec{r}\,'(t)|\, dt.$$

$$\int_C f\, ds = \int_a^b f(\vec{r}(t)) \cdot \underline{|\vec{r}\,'(t)|\, dt}$$

<u>EX</u>: line integral of $\boxed{f(x,y,z) = x+y+z}$ over the line segment from $(3,1,4)$ to $(2,-1,1)$



First parametrize the line segment.

$$\vec{r}(t) = \vec{r}_0 + t\cdot\vec{v} = \langle 3,1,4\rangle + t\cdot\langle -1,-2,-3\rangle, \quad \underline{0 \le t \le 1}$$

$$= \langle 3-t, 1-2t, 4-3t\rangle$$

$$\vec{v} = \langle 2,-1,1\rangle - \langle 3,1,4\rangle = \langle -1,-2,-3\rangle$$

$$\vec{r}\,'(t) = \langle -1,-2,-3\rangle \qquad |\vec{r}\,'(t)| = \sqrt{1+4+9} = \sqrt{14}$$

$$\int_C \overset{x+y+z}{\overset{\parallel}{f}}\, ds = \int_0^1 (3-t+1-2t+4-3t)\sqrt{14}\, dt = \sqrt{14}\int_0^1 (8-6t)\,dt$$

$$= \sqrt{14}\cdot(8t-3t^2)\Big|_0^1 = \sqrt{14}\cdot(5) = 5\sqrt{14}.$$

<u>EX</u>: $f(x,y,z) = \underline{x+\sqrt{y}-z^4}$. $\quad \underline{\vec{r}(t) = t\vec{i} + t^2\vec{j}}$, $\quad \underline{0 \le t \le 1}$.

$$\vec{r}\,'(t) = \vec{i} + 2t\cdot\vec{j}, \quad \underline{|\vec{r}\,'(t)| = \sqrt{1+(2t)^2} = \sqrt{1+4t^2}}$$

$$\int_C f\, ds = \int_0^1 (t+\sqrt{t^2}-0^4)\sqrt{1+4t^2}\,dt = \int_0^1 2t\cdot\sqrt{1+4t^2}\,dt$$

$u = 1+4t^2$

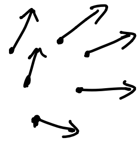
$du = 8t\cdot dt$

$2t\cdot dt = \frac{du}{4}$

$$= \int_1^5 \sqrt{u}\,\frac{du}{4}$$

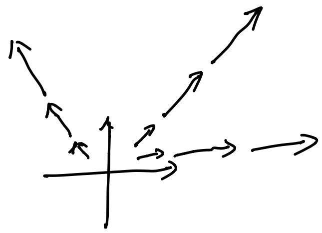
$$\frac{1}{6}(5\sqrt{5}-1) = \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^5$$

16.2 <u>Vector fields</u> and line integrals.



3d: $\vec{F}(x,y,z) = \underline{M(x,y,z)}\,\vec{i} + \underline{N(x,y,z)}\,\vec{j} + \underline{P(x,y,z)}\,\vec{k}$.

2d: $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$.



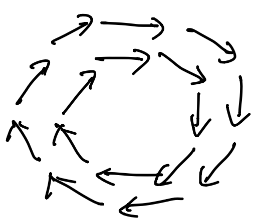
Ex: radial vector field: $\vec{F} = x\vec{i} + y\vec{j}$

spin field of unit vector

$$\underline{\vec{F} = \frac{-x\vec{i} + y\vec{j}}{\sqrt{x^2+y^2}}}$$

$$\vec{F} = \frac{x\vec{i} - y\vec{j}}{\sqrt{x^2+y^2}}$$

not defined at 0.



Ex: Gradient vector fields.

$f = f(x,y,z)$. $\nabla f\,(x,y,z) = \langle f_x, f_y, f_z \rangle \underline{(x,y,z)}$

$\uparrow$ grad. vector field of f.

$\nabla f \perp$ level surface.

velocity fields in physics.
force

$\underline{C:\ \vec{r} = \vec{r}(t),\ a \le t \le b.}$

Def: $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ with $M = M(x,y,z)$, $N = N(x,y,z)$, $P = P(x,y,z)$ continuous.

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt} = \int_{s_{min}}^{s_{max}} \vec{F}\cdot\vec{r}\,'(s)\,ds$$

$\vec{F} = \vec{F}(x(t), y(t), z(t))$ $\qquad \vec{r}\,'(s) = \vec{T}(s)$ unit tangent vector

$d\vec{r} = \vec{r}\,'(t)\,dt = \underline{\vec{r}\,'(s)\,ds}$ $\qquad (|\vec{T}(s)| \equiv 1,\ s_{min} \le s \le s_{max})$

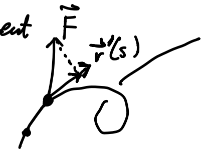
$\uparrow$ arclength parametrization

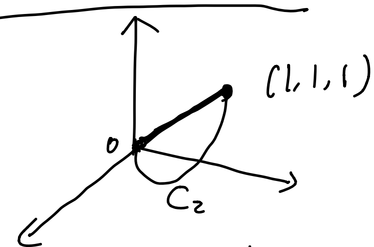
Work done by the force:

change of displacement

$$\int_C \vec{F}\cdot d\vec{r} = \int_{s_{min}}^{s_{max}} \underline{\vec{F}\cdot\vec{r}\,'(s)}\, ds$$

Force in the direction of movement





Ex: $\vec{F} = \sqrt{z}\,\vec{i} - 2x\,\vec{j} + 3\sqrt{y}\,\vec{k}$

C: $\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k}$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 1, 1\rangle$. $\quad \vec{F}\cdot\vec{r}\,' = \sqrt{z}\cdot 1 - 2x\cdot 1 + 3\sqrt{y}\Big|_{\vec{r}=\vec{r}(t)} = \sqrt{t} - 2t + 3\sqrt{t}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \underbrace{(\sqrt{t} - 2t + 3\sqrt{t})}_{\vec{F}\cdot\vec{r}\,'} dt = \int_0^1 (4\sqrt{t} - 2t)\, dt$$

$$= \left(4\cdot 2\cdot t^{\frac{1}{2}} - t^2\right)\Big|_0^1 = 4\times 2\times 1 - 1 - 0 = 7.$$

C_2: $\vec{r}(t) = \underline{t}\,\vec{i} + \underline{t^2}\,\vec{j} + \underline{t^4}\,\vec{k}$. $\quad 0 \le t \le 1$

$\vec{r}\,'(t) = \vec{i} + 2t\,\vec{j} + 4\cdot t^3\,\vec{k}$

$$\vec{F}\cdot\vec{r}\,'(t) = \sqrt{z}\cdot 1 + (-2x)\cdot 2t + 3\sqrt{y}\cdot 4t^3\Big|_{\vec{r}=\vec{r}(t)}$$

$$= \sqrt{t^4}\cdot 1 + (-2t)\cdot 2t + 3\sqrt{t^2}\cdot 4t^3$$

$$= t^2 - 4t^2 + 12t^4 = 12t^4 - 3t^2$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot\vec{r}\,'(t)\, dt = \int_0^1 (12t^4 - 3t^2)\, dt$$

$$= \left(\frac{12}{5}t^5 - t^3\right)\Big|_0^1 = \frac{12}{5} - 1 = \frac{7}{5}.$$

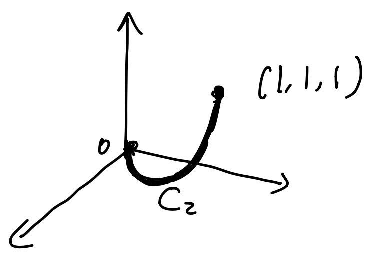
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \underbrace{(M\,dx + N\,dy + P\,dz)}_{\text{line integrals of differential forms.}}$$

$$\vec{F} \cdot d\vec{r} = (M\vec{i} + N\vec{j} + P\vec{k}) \cdot d(x\vec{i} + y\vec{j} + z\vec{k}) = \langle M, N, P \rangle \cdot \langle dx, dy, dz \rangle$$

$$\int_C M\,dx \overset{\vec{r}=\vec{r}(t)}{=\!=} \int_a^b M(x(t), y(t), z(t))\, x'(t)\,dt$$

$$\int_C N\,dy = \int_a^b N(x(t), y(t), z(t))\, y'(t)\,dt.$$

$$M\vec{i} + N\vec{j} + P\vec{k}$$



Ex: $\vec{F} = \sqrt{z}\,\vec{i} - 2x\,\vec{j} + 3\sqrt{y}\,\vec{k}$

$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^4\vec{k}$. $0 \le t \le 1$

$$\boxed{\int_{C_2} M\,dx - N\,dy}$$

$$= \int_a^b M(x(t), y(t), z(t))\, x'(t)\,dt - N(x(t), y(t), z(t))\, y'(t)\,dt$$

$$= \int_a^b (M \cdot x' - N \cdot y')\,dt$$

$$= \int_0^1 \left(\sqrt{t^4} \cdot 1 - (-2t) \cdot 2t\right) dt = \int_0^1 \underbrace{(t^2 + 4t^2)}_{5t^2} dt = \frac{5}{3} t^3 \Big|_0^1$$

$$= \frac{5}{3}.$$

Application to fluid flows:

$\vec{F}$: velocity field of fluid

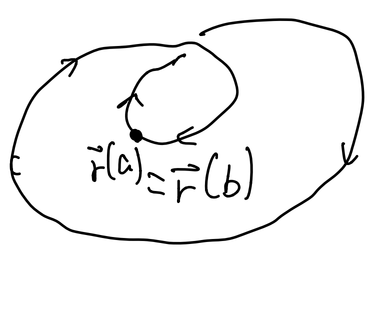
$\vec{r}(t)$: parametrization of a curve.

Def: the flow of $\vec{F}$ along the curve from $\vec{r}(a)$ to $\vec{r}(b)$ is

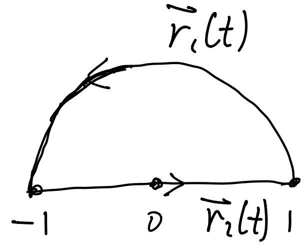
$$\text{Flow} = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \underbrace{\vec{T}}_{= \frac{d\vec{r}}{dt} \text{ unit tangent vector}} ds$$

$$= \int_C \vec{F} \cdot \vec{r}\,'(t)\, dt$$

If $\vec{r}(a) = \vec{r}(b)$, then the flow is called the circulation along the curve.



Ex: $\vec{F} = -3y\,\vec{i} + 3x\,\vec{j}$



$\vec{r}_1(t) = \cos t \cdot \vec{i} + \sin t\,\vec{j}\,,\ 0 \le t \le \pi$

$\vec{r}_2(t) = \underbrace{t\,\vec{i}}_{= \langle t, 0\rangle}\,,\ -1 \le t \le 1$

Circulation: $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

For C_1: $\vec{F} \cdot \vec{r}_1{}'(t) = \langle -3 \cdot \sin t, 3\cos t\rangle \cdot \langle -\sin t, \cos t\rangle$

$= 3 \cdot \sin^2 t + 3 \cdot \cos^2 t = 3$.

$\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F} \cdot \vec{r}\,'(t)\,dt = \int_0^{\pi} 3\,dt = 3\pi$

For C_2: $\underbrace{\vec{F}}_{\langle -3y, 3x\rangle} \cdot \underbrace{\vec{r}_2{}'(t)}_{\langle 1, 0\rangle} = \underbrace{\langle -3 \cdot 0, 3 \cdot t\rangle}_{\langle 0, 3t\rangle} \cdot \langle 1, 0\rangle = 0$

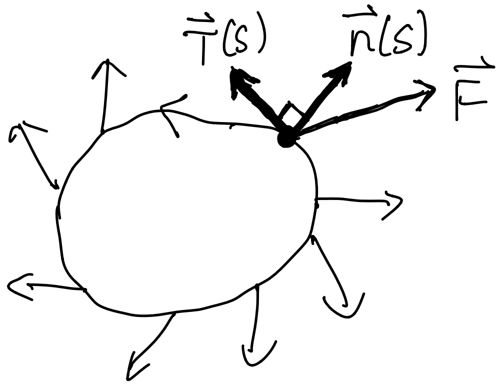
$\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \vec{F} \cdot \vec{r}_2{}'(t)\,dt = \int_{-1}^{1} 0 \cdot dt = 0$.

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 3\pi}$$

The flux of $\vec{F}$ across C

$\vec{F}$: vector field on plane

C: a closed curve in the plane



$$\boxed{\int_C \vec{F} \bullet \underbrace{\vec{n}(s)}_{\text{unit normal vector}} \, ds}$$

flux

circulation: $\int_C \vec{F} \bullet \vec{T}(s)\, ds$

rotate $\vec{T}(s)$ by 90° clockwise

$$\vec{T}(s) = \frac{d\vec{r}}{ds} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0 \right\rangle \quad \longrightarrow \quad \vec{n}(s) = \vec{T} \times \vec{k} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0 \right\rangle \times \langle 0, 0, 1 \rangle$$

$$\boxed{\vec{n}(s) = \left\langle \frac{dy}{ds}, -\frac{dx}{ds}, 0 \right\rangle}$$

$$\oint_C \vec{F} \bullet \vec{n}(s)\, ds = \oint_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle ds$$

$$= \oint_C \underline{M\,dy - N\,dx} \quad \text{flux across } C.$$