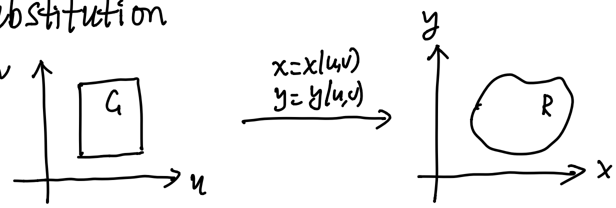
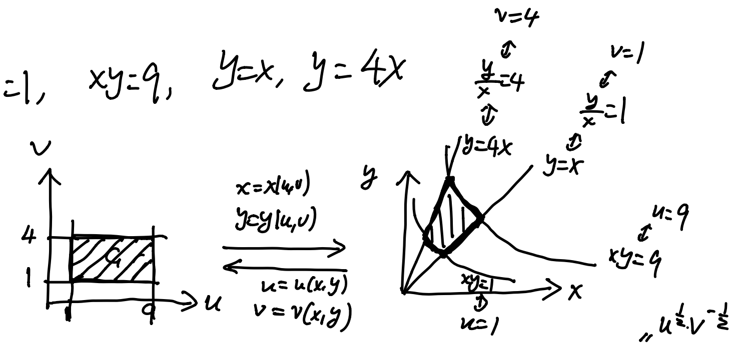
Substitution



$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v),\, y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Ex: R: bounded by $xy=1$, $xy=9$, $y=x$, $y=4x$



$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dx\,dy$$

$$\parallel$$

$$\iint_G (\sqrt{v}+\sqrt{u})\cdot\left|\frac{1}{2}\frac{1}{v}\right|\cdot du\,dv$$

$$\parallel$$

$$\frac{1}{2}\int_1^4\int_1^9 \left(v^{-\frac{1}{2}}+u^{\frac{1}{2}}v^{-1}\right)du\,dv$$

$$\parallel$$

$$\frac{1}{2}\cdot\int_1^4 \left(v^{-\frac{1}{2}}u+v^{-1}\cdot\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^9 dv$$

$$\parallel$$

$$\frac{1}{2}\cdot\int_1^4 \left(8\cdot v^{-\frac{1}{2}}+\frac{2}{3}\cdot v^{-1}\cdot(27-1)\right)dv$$

$$\parallel$$

$$4\cdot\int_1^4 v^{-\frac{1}{2}}dv+\frac{26}{3}\int_1^4 v^{-1}dv$$

$$\parallel$$

$$4\cdot\left(2\cdot v^{\frac{1}{2}}\Big|_1^4\right)+\frac{26}{3}\ln v\Big|_1^4$$

$$\parallel$$

$$8\times(2-1)+\frac{26}{3}\ln 4 = 8+\frac{26}{3}\ln 4.$$

$$\parallel$$

$$8+\frac{52}{3}\ln 2$$

$$\boxed{u=xy,\ v=\frac{y}{x}} \Leftrightarrow \frac{u}{v}=x^2 \Leftrightarrow x=\sqrt{\frac{u}{v}} = u^{\frac{1}{2}}\cdot v^{-\frac{1}{2}}$$

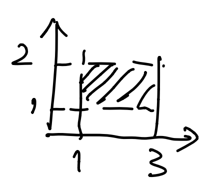
$$uv=y^2 \Leftrightarrow y=\sqrt{uv}=u^{\frac{1}{2}}\cdot v^{\frac{1}{2}}$$

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \underbrace{\frac{1}{\sqrt{v}}\cdot\frac{1}{2}\frac{1}{\sqrt{u}}}_{\frac{1}{2}v^{-\frac{1}{2}}\cdot u^{-\frac{1}{2}}} & \sqrt{u}\cdot\left(-\frac{1}{2}\right)\cdot v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & u^{\frac{1}{2}}\cdot\frac{1}{2}v^{-\frac{1}{2}} \end{vmatrix}$$

$$=\frac{1}{4}v^{-1}+\frac{1}{4}\cdot v^{-1}=\frac{1}{2}v^{-1}$$

$$u=\sqrt{xy}\,,\quad v=\sqrt{\frac{y}{x}}$$

$$\rightsquigarrow\ x=\frac{u}{v},\quad y=u\cdot v.$$



$$\boxed{\iint_{G'} (v+u)\cdot\frac{2u}{v}\,du\,dv}$$

$$J(u,v)=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2\cdot u}{v}$$

$$\iint_{G'} (v+u)\frac{2u}{v}\,du\,dv = \int_1^2\int_1^3 \left(2u+\frac{2u^2}{v}\right)du\,dv$$

$$=\int_1^2 \left(u^2+\frac{2}{3}\frac{u^3}{v}\right)\Big|_1^3\,dv$$

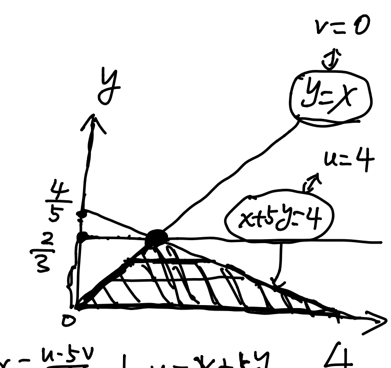
$$=\int_1^2\left(9+\frac{18}{v}-1-\frac{2}{3}\frac{1}{v}\right)dv$$

$$=\left(8v+\frac{52}{3}\ln v\right)\Big|_1^2 = 8+\frac{52}{3}\ln 2$$

Ex: $\displaystyle\int_0^{\frac{2}{3}}\int_y^{4-5y} (x+5y)\,e^{y-x}\,dx\,dy$

$0\le y\le \frac{2}{3}$, $\quad y\le x\le 4-5y$

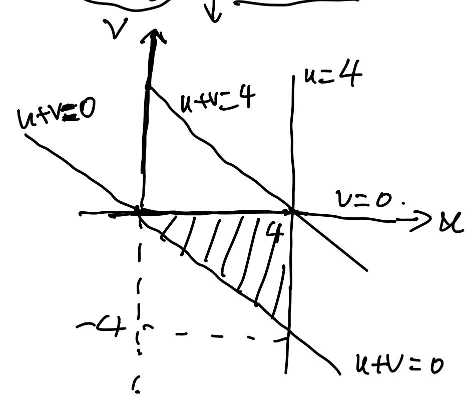
$y\le x$, $\quad x+5y\le 4$



$x=\frac{u-5v}{6}$ $\quad u=x+5y$

$y=\frac{u+v}{6}$ $\quad v=y-x$

$$\boxed{\int_0^4\int_{-u}^0 u\cdot e^v\,\frac{1}{6}\,dv\,du}$$



$y=\frac{2}{3}=\frac{u+v}{6}\iff u+v=4$

$$\begin{cases} x - y = 0 \\ x + 5y = 4 \end{cases} \Rightarrow 6y = 4 \Rightarrow y = \frac{4}{6} = \frac{2}{3} = x$$

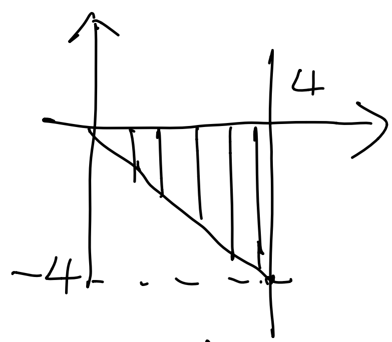
$$x = y - v = \frac{u+v}{6} - v = \frac{u-5v}{6}$$

$$u - 5y = u - \frac{5u+5v}{6} = \frac{u-5v}{6}$$

$$y = \frac{u+v}{6}$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{6} & \frac{1}{6} \end{vmatrix} = \frac{1}{36} + \frac{5}{36} = \frac{1}{6}$$

$$\int_0^4 \int_{-u}^0 u \cdot e^v \cdot \frac{1}{6} \, dv \, du$$



$$= \frac{1}{6} \cdot \int_0^4 u \cdot e^v \Big|_{-u}^0 du = \frac{1}{6} \int_0^4 u \cdot (1 - e^{-u}) \, du$$

$$= \frac{1}{6} \cdot \left(\frac{1}{2} u^2 - u \cdot e^{-u} - e^{-u} \right) \Big|_0^4$$

$$= \frac{1}{6} \cdot \left(8 - 4 \cdot e^{-4} - e^{-4} - (-1) \right) = \boxed{\frac{1}{6} \cdot (9 - 5 \cdot e^{-4})}$$

$$\int \underline{u \cdot e^{-u} du} = -\int u \cdot d(e^{-u}) = -u \cdot e^{-u} + \int e^{-u} du$$

$$= \boxed{-ue^{-u} - e^{-u}}$$

$$(-ue^{-u} - e^{-u})' = -e^{-u} + ue^{-u} + e^{-u}$$

$$= ue^{-u}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_G f(\ \ )\cdot|J(u,v,w)|\,du\,dv\,dw$$

$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{cases} x = \rho\cdot\sin\phi\cdot\cos\theta = x(\rho,\phi,\theta) \\ y = \rho\cdot\sin\phi\cdot\sin\theta = y(\rho,\phi,\theta) \\ z = \rho\cdot\cos\phi = z(\rho,\phi,\theta) \end{cases}$$

$$J(\rho,\phi,\theta) = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ & & \\ & & \end{vmatrix}$$

$$\begin{vmatrix} \sin\phi\cdot\cos\theta & \rho\cos\phi\cos\theta & -\rho\cdot\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cdot\cos\phi\sin\theta & \rho\cdot\sin\phi\cos\theta \\ \cos\phi & -\rho\cdot\sin\phi & 0 \end{vmatrix} = \cos\phi\cdot\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cdot\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$+\rho\cdot\sin\phi\cdot\begin{vmatrix} \sin\phi\cdot\cos\theta & -\rho\cdot\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= (\cos\phi)\left(\rho^2 \cdot \cos\phi \sin\phi \cos^2\theta + \rho^2 \sin\phi \cos\phi \sin^2\theta\right)$$

$$+ \rho \cdot \sin\phi \cdot \left(\rho \cdot \sin^2\phi \cos^2\theta + \rho \cdot \sin^2\phi \sin^2\theta\right)$$

$$= \cos\phi \cdot \rho^2 \cdot \sin\phi \cdot \cos\phi + \rho^2 \cdot \sin\phi \cdot \sin^2\phi$$

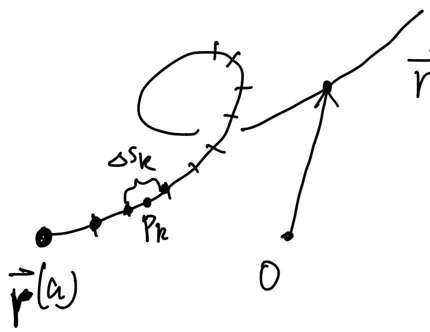
$$= \rho^2 \sin\phi \left(\cos^2\phi + \sin^2\phi\right) = \boxed{\rho^2 \cdot \sin\phi}$$

$$\iiint \underset{\substack{\parallel \\ dx\,dy\,dz}}{dV} = \iiint \underset{\substack{\parallel \\ \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi}}{\boxed{\rho^2 \sin\phi}}\, d\rho\, d\phi\, d\theta$$

$$\boxed{\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r}$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

16.1 line integrals of scalar functions.



$\vec{r}(t) = x(t)\vec{i} + y(t)\cdot\vec{j} + z(t)\vec{k}$. $a \le t \le b$.

$f = f(x, y, z)$.

$t \mapsto f(x(t), y(t), z(t))$

line integral of f over C:

$$\boxed{\int_C f(x,y,z)\,ds = \lim_{\|P\| \to 0} \sum_k f(P_k)\,\Delta s_k}$$

arclength parameter

(f = density per unit length $\rightsquigarrow$ mass of the curve)

$$s = s(t) = \int_a^t |\vec{r}'(\tau)|\,d\tau, \quad \boxed{ds = s'(t)\,dt = |\vec{r}'(t)|\,dt.}$$

velocity

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,|\vec{r}'(t)|\,dt.}$$

$C = \{\vec{r}(t);\ a \le t \le b\}$ does NOT depend on the parametrization $\vec{r} = \vec{r}(t)$. $t = t(u)$.

$$\boxed{\begin{array}{c} t = t(u) \\ \vec{r} = \vec{r}(t) = \vec{r}(t(u)) \\ \quad\quad \shortparallel \\ \quad\quad \vec{r}(u) \end{array}} = \int_c^d f(x(u), y(u), z(u)) \, |\vec{r}\,'(u)| \, du$$

$$ds = |\vec{r}\,'(t)| \, dt = |\vec{r}\,'(u)| \, du$$