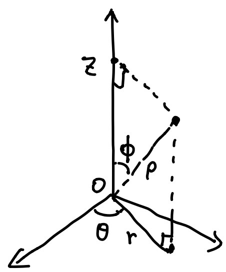


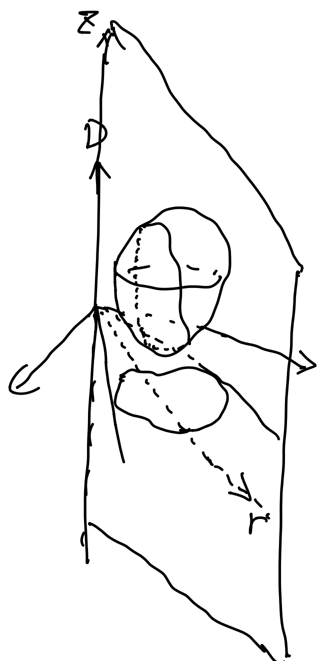

$$\begin{cases} x = r\cos\theta = \rho\cdot\sin\phi\cdot\cos\theta \\ y = r\sin\theta = \rho\cdot\sin\phi\cdot\sin\theta \\ z = z = \rho\cdot\cos\phi \end{cases}$$

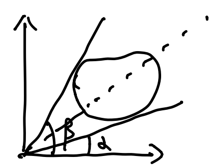
$r \geq 0,\ \rho \geq 0$

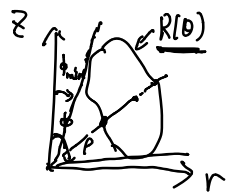
$0 \leq \phi \leq \pi$

$0 \leq \theta \leq 2\pi$

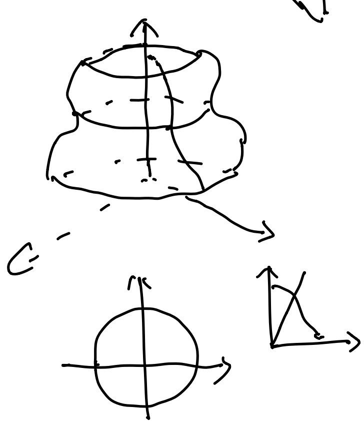
$$\iiint f(x,y,z)\,dV = \int_{\alpha}^{\beta}\int_{\phi_{min}(\theta)}^{\phi_{max}(\theta)}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\boxed{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$$

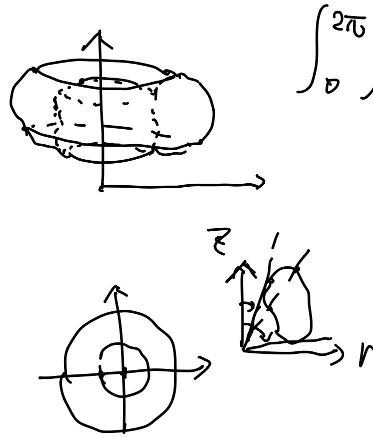


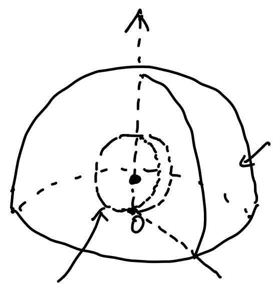



$$\int_{\phi_{min}(\theta)}^{\phi_{max}(\theta)}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} d\rho\ \ d\phi$$






$$\int_{0}^{2\pi}\int_{\phi_{min}}^{\phi_{max}}\int_{g_1(\phi)}^{g_2(\phi)} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$




$x^2+y^2+(z-3)^2=3^2$

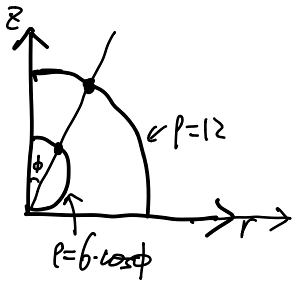
center $(0,0,3)$, radius $=3$

$\underbrace{x^2+y^2+z^2}_{=\rho^2}-6z+\cancel{9}=\cancel{9}\ 0$

$\rho^2=6\cdot z=6\cdot\rho\cos\phi$

$\boxed{\rho=6\cdot\cos\phi}$

$$\iiint_D dV=\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\int_{6\cdot\cos\phi}^{12}\rho^2\sin\phi\,d\rho\,d\phi\right)d\theta$$




$$=2\pi\cdot\int_0^{\frac{\pi}{2}}\left(\frac{\rho^3}{3}\Bigg|_{6\cos\phi}^{12}\right)\cdot\sin\phi\,d\phi$$

$$=2\pi\cdot\int_0^{\frac{\pi}{2}}\cdot\left(\frac{12^3}{3}-\frac{6^3\cos^3\phi}{3}\right)\sin\phi\,d\phi$$

$$=2\pi\cdot\left[4\times12^2\cdot(-\cos\phi)\Big|_0^{\frac{\pi}{2}}+2\times36\cdot\int_0^{\frac{\pi}{2}}\cos^3\phi\,d\cos\phi\right]$$

$$=2\pi\cdot\left[4\times144\times1+72\cdot\frac{1}{4}\cdot\cos^4\phi\Big|_0^{\frac{\pi}{2}}\right]$$

$$=2\pi\cdot[576+18\times(-1)]$$

$$=\underline{2\pi\times(558)}$$

$$\overset{\checkmark}{=}2\pi\cdot(4\times144-18)$$

$$=2\pi\times4\times12^2-4\pi\times9$$

Check by using

$\boxed{vol(B_R)=\frac{4}{3}\pi\cdot R^3}$ :

$$vol(D)=\frac{1}{2}Vol(B_{12})-Vol(B_3)$$

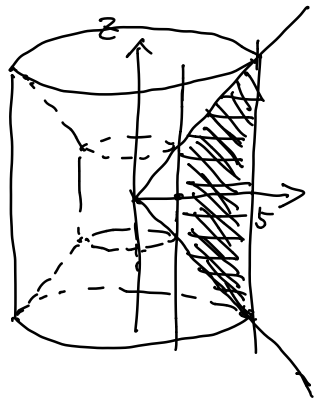
$$=\underline{\frac{1}{2}\cdot\frac{4}{3}\pi\times12^3-\frac{4}{3}\pi\cdot3^3}=2\pi\times4\times12^2-4\pi\times9$$

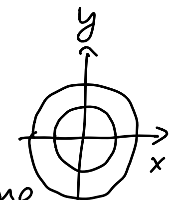
Ex: Find the volume of the solid cut out from: $1 \le x^2+y^2 \le 5$ by the cones $z = \pm\sqrt{4x^2+4y^2}$

$1 \le r^2 \le 5 \Leftrightarrow 1 \le r \le \sqrt{5}$

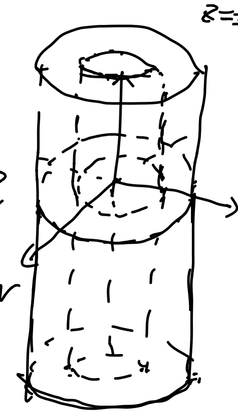
$z^2 = 4x^2+4y^2$

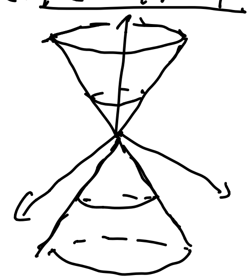
$z = \pm 2r \Leftrightarrow \boxed{z^2 = 4r^2.}$



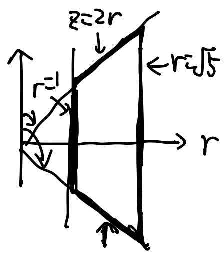

cylinder





$$\iiint dV = \int_0^{2\pi}\left(\int_1^{\sqrt{5}}\int_{-2r}^{2r} r\,dz\,dr\right)d\theta$$

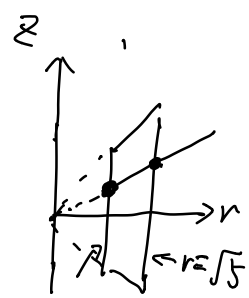
$$= 2\pi\cdot\int_1^{\sqrt{5}} r\cdot 4r\,dr = 8\pi\cdot\frac{r^3}{3}\Big|_1^{\sqrt{5}} = \frac{8\pi}{3}\cdot(5\sqrt{5}-1).$$




Spherical coordinates

$$= \boxed{\int_0^{2\pi}\int_{\underset{\phi_{min}}{\tan^{-1}(\frac{1}{2})}}^{\overset{\pi-\tan^{-1}(\frac{1}{2})}{\phi_{max}}}\int_{\frac{1}{\sin\phi}}^{\frac{\sqrt{5}}{\sin\phi}} (\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta}$$

$(\tan\phi)_{min}^{-1} = 2 \Rightarrow \phi_{min} = \tan^{-1}\left(\frac{1}{2}\right)$

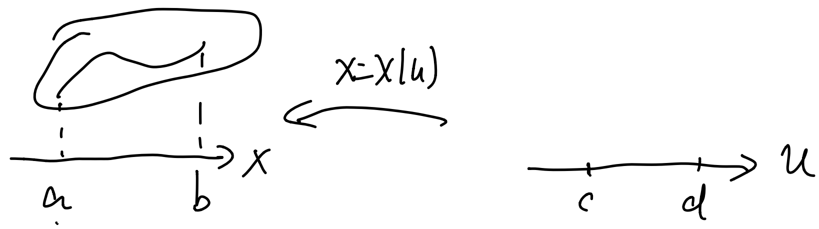



$r=1 = \rho\cdot\sin\phi.$

substitution

$$\boxed{dx = x'(u)\,du}$$
$$x = x(u)$$

$$\int_a^b f(x)\,dx \;=\!=\; \int_c^d f(x(u))\cdot x'(u)\,du$$




$x(c) = a \qquad x(d) = b$

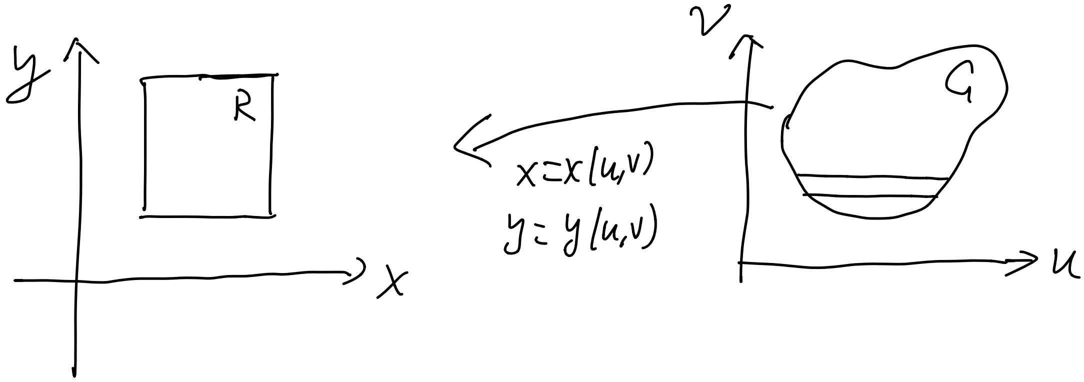
Ex:

$$dx = -\sin\phi\, d\phi$$
$$x = \cos\phi$$

simple $\rightarrow \int_1^0 (-x^3)\,dx \;=\!=\; \int_0^{\frac{\pi}{2}} \cos^3\phi \cdot \sin\phi\, d\phi$

$$\| \quad \int_0^1 x^3\,dx$$

$$\boxed{\iint_R f(x,y)\,\underset{dV}{dx\,dy} \;\underset{y=y(u,v)}{\overset{x=x(u,v)}{=\!=}}\; \iint_G f(x(u,v),\, y(u,v))\,|J(u,v)|\,du\,dv}$$



$x = x(u,v) \qquad dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$

$y = y(u,v) \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$

Change of orientation

$\downarrow$

$du\,dv = -dv\,du$

$$\underline{dx\,dy} = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right).$$

$$= \cancel{\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}du\,du} + \boxed{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}du\,dv + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dv\,du} + \cancel{\frac{\partial x}{\partial v}\frac{\partial y}{\partial v}dv\,dv}$$

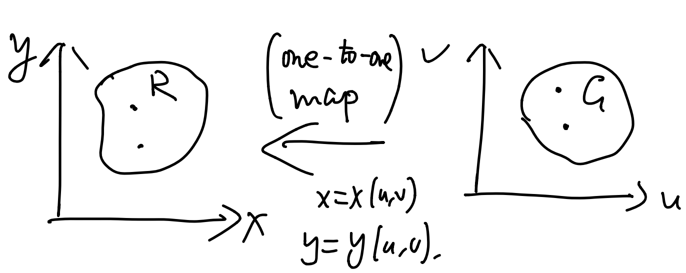
$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\,dv$$

$$\underline{dx\,dy} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv \qquad dx = x'(u)\,du$$

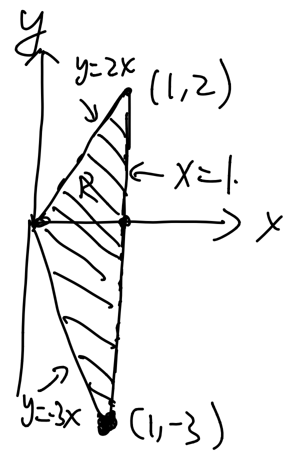
$$= J(u,v)\,du\,dv \qquad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v), y(u,v))\,\underline{|J(u,v)|}\,du\,dv$$

$$\| \quad \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right|$$



Ex:

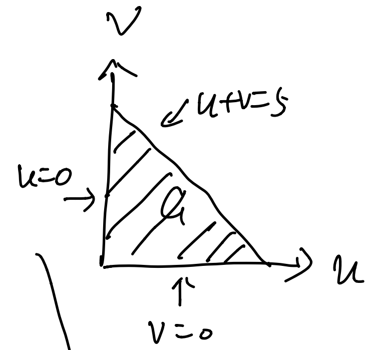



$$\begin{cases} u = 2x - y \\ v = 3x + y \end{cases}$$

$$\longrightarrow$$

$$x = \frac{u+v}{5}$$

$$y = 2x - u = \frac{2}{5}(u+v) - u = -\frac{3}{5}u + \frac{2}{5}v$$




$x = 1 \iff u + v = 5$

$y = 2x \iff u = 0$

$\Updownarrow$

$2x - y = 0 \iff v = 0$

$y = -3x$

$\Updownarrow$

$3x + y = 0$

$$\boxed{\begin{gathered} \iint_G du\,dv = \frac{5 \times 5}{2} = \frac{25}{2} \\ \cancel{\parallel} \\ \iint_R dx\,dy \end{gathered}}$$

$$\frac{5 \times 1}{2} = \frac{5}{2}$$

$$\parallel$$

$$\iint_R \underline{dx\,dy} = \iint_G \underbrace{\overbrace{|J(u,v)|}^{\frac{1}{5}}}_{\frac{1}{5} \times \frac{25}{2} = \frac{5}{2}} \underline{du\,dv}$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{vmatrix} = \frac{2}{25} - \frac{-3}{25} = \frac{5}{25} = \underline{\frac{1}{5}}$$