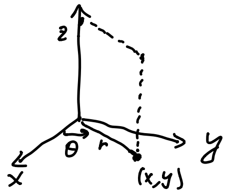
Cylindrical coordinates. (r, θ, z)

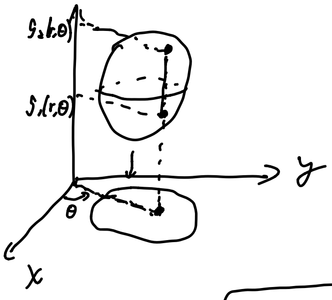
$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z. \end{cases}$$

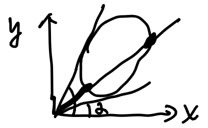


$$\iiint_D f(x,y,z)\,\underline{dV}$$

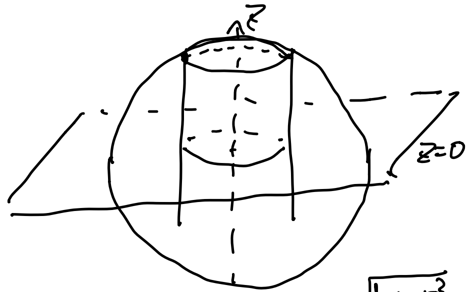
$$||$$

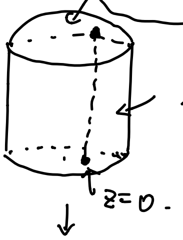
$$\int_{\partial}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cdot\cos\theta, r\sin\theta, z)\,dz\,\overset{\overset{dxdy}{''}}{\underline{r dr d\theta}}$$





<u>Ex</u>:

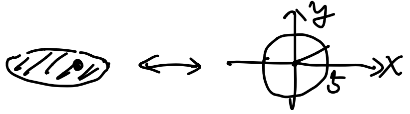




$$\boxed{x^2+y^2+z^2=100} \leftrightarrow z = \sqrt{100-(x^2+y^2)} = \sqrt{100-r^2}$$

$$x^2+y^2=25$$

$$\iiint dV = \int_0^{2\pi}\int_0^5\int_0^{\sqrt{100-r^2}} \underline{dz\,\textcircled{r}dr\,d\theta}$$



$$\frac{1}{2}dr^2$$

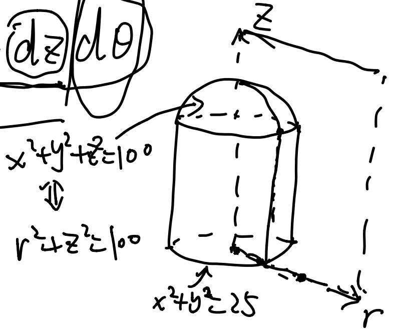
$$= \int_0^{2\pi}\int_0^5 \sqrt{100-r^2}\,\overset{||}{\boxed{rdr}}\,d\theta \qquad \underline{u=100-r^2}, \quad du=-2rdr \quad rdr = -\frac{du}{2}$$

$$= \int_0^{2\pi}\int_{100}^{25} \sqrt{u}\cdot\left(-\frac{du}{2}\right)d\theta$$

$$= \int_0^{2\pi}\int_{25}^{100} \frac{1}{2}u^{\frac{1}{2}}du\,d\theta = \int_0^{2\pi}\underline{\left(\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{\frac{3}{2}}\Big|_{25}^{100}\right)}d\theta$$

$$= 2\pi\cdot\frac{1}{3}\cdot\left(100^{\frac{3}{2}} - \underset{\underset{(3\times 25)^{\frac{3}{2}} = (\sqrt{3}\cdot 5)^3}{||}}{\boxed{75^{\frac{3}{2}}}}\right) = \frac{2\pi}{3}\cdot\left(1000 - (125\times 3)\sqrt{3}\right).$$

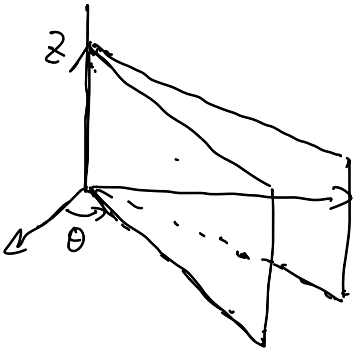
$$\iiint dV = \int_0^{2\pi} \boxed{\int \int \underline{\boxed{r\,dr\,\boxed{dz}}}}\,\boxed{d\theta}$$

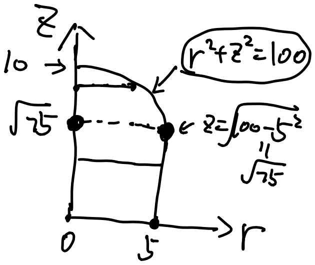


$$\rotatebox{90}{=}$$

$$\iiint \underline{dz\cdot r\,dr\,d\theta}$$

$\theta = \theta_0$





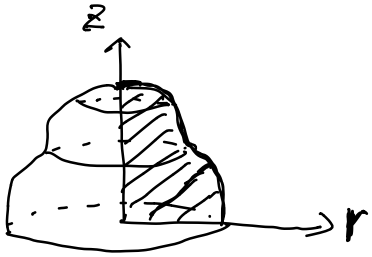
$$\iint_R r\,dr\,dz = \left(\int_0^{\sqrt{75}}\int_0^5 r\,dr\,dz + \int_{\sqrt{75}}^{10}\int_0^{\sqrt{100-z^2}} r\,dr\,dz\right)$$

$$\iiint dV = \int_0^{2\pi}\left(\qquad \right) dz$$

rotationally symmetric around z-axis ⇝ use cylindrical coordinate.

use the slice of intersection of solid with $\theta=\theta_0$.

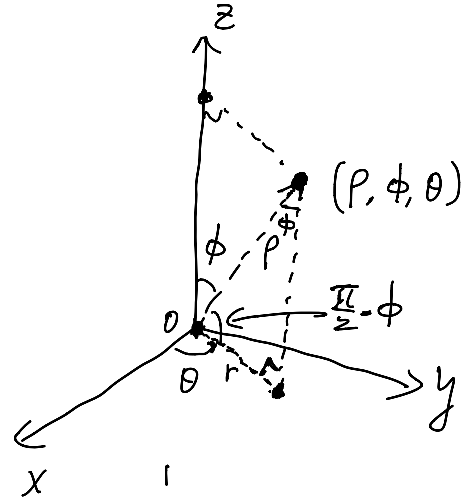
$dx\,dy \to \boxed{r}\,dr\,d\theta$



Spherical coordinates

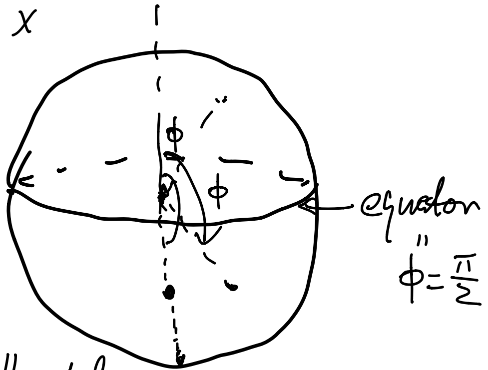
$$\boxed{r = \rho \cdot \sin\phi}$$

$$\begin{cases} x = r\cdot\cos\theta = \rho\cdot\sin\phi\cdot\cos\theta \\ y = r\cdot\sin\theta = \rho\cdot\sin\phi\cdot\sin\theta \\ z = \rho\cdot\cos\phi \end{cases}$$



$\rho \geq 0$, $0 \leq \phi \leq \pi$. $0 \leq \theta \leq 2\pi$

↑ latitude ↑ meridian



$$\iiint f(x,y,z)\,\boxed{dV}$$ ← volume differential w.r.t. infitesimal change of (ρ,ϕ,θ)

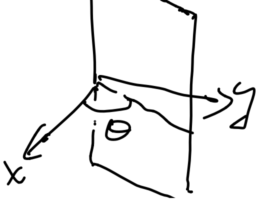
$$= \int_\alpha^\beta \int \int f(\underbrace{\rho\sin\phi\cos\theta}_{x}, \underbrace{\rho\sin\phi\sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z})\,\boxed{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

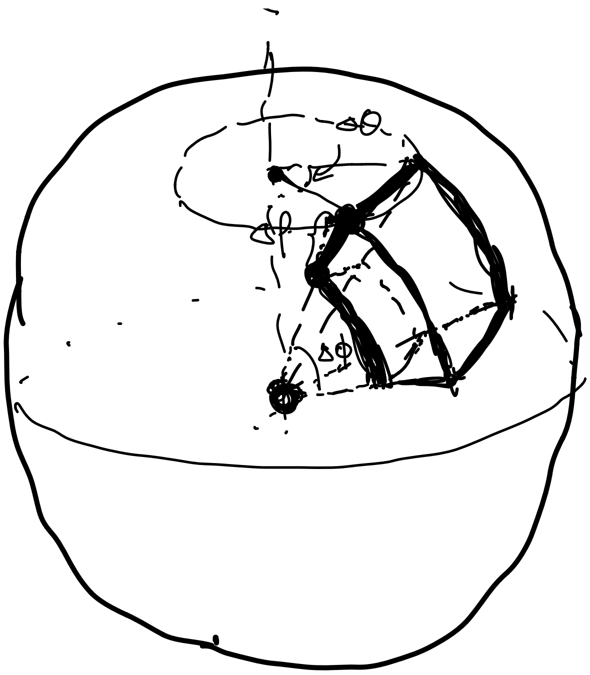
$\rho = \rho_0$: sphere of radius ρ

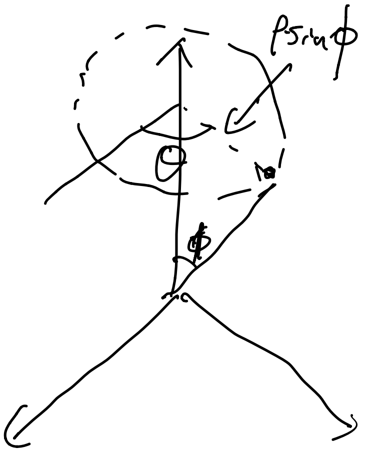
← $\phi = \phi_0$ a cone

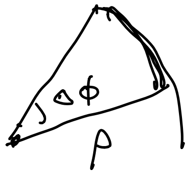
$\phi = \phi_0$:

$\theta = \theta_0$:









$$\Delta\rho \cdot (\rho \cdot \Delta\phi)(\rho \sin\phi)\Delta\theta$$

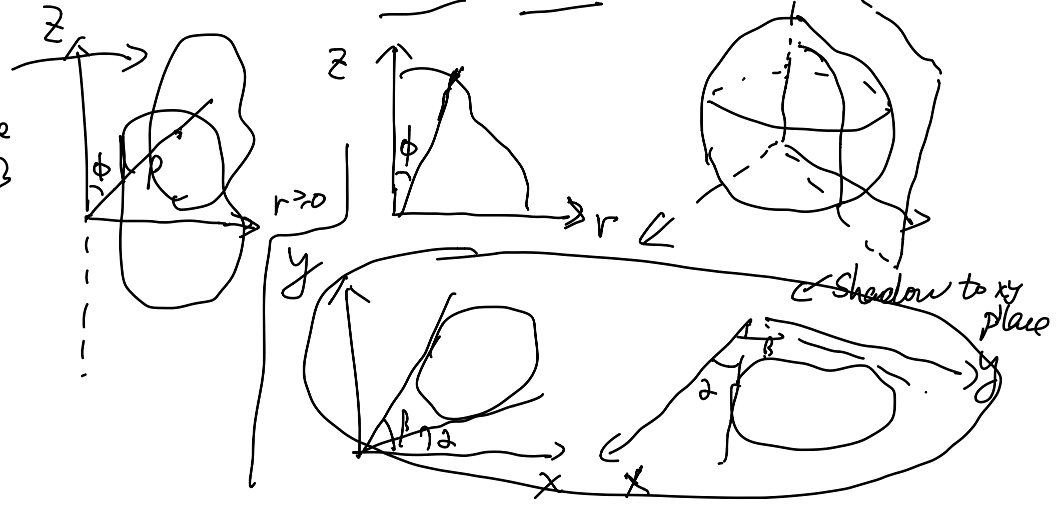
$$\downarrow$$

$$\rho^2 \cdot \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\iiint dV = \int_{\alpha}^{\beta} \int_{\underline{f_1(\phi)}}^{\underline{f_2(\phi)}} \int_{\underline{g_1(\phi,\theta)}}^{\underline{g_2(\phi,\theta)}} \boxed{\rho^2 \cdot \sin\phi \, d\rho \, d\phi} \, d\theta .$$

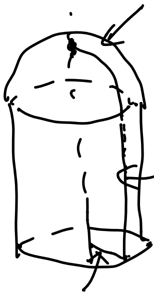
slice cut out by the plane

$\alpha \leq \theta = \theta_0 \leq \beta$



$\rho = r^2 + z^2$

Ex:



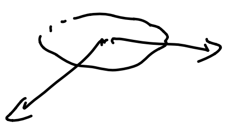
$\underbrace{x^2+y^2+z^2}_{\rho^2}=100 \iff \rho=10.$

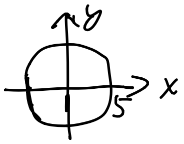
$\underbrace{x^2+y^2}_{r^2=\rho^2\cdot\sin^2\phi}=25. \iff \rho\cdot\sin\phi=5.$

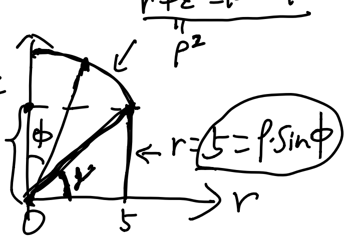
$$\begin{cases} x=\rho\cdot\sin\phi\cos\theta \\ y=\rho\cdot\sin\phi\sin\theta \\ z=\rho\cdot\cos\phi \end{cases}$$

rotationally symmetric

$z=0 \iff \phi=\frac{\pi}{2}$







$\underbrace{r^2+z^2}_{\rho^2}=100=10^2$

$5\sqrt{3}=\sqrt{75}$

$r=\boxed{5=\rho\cdot\sin\phi}$

$\tan\gamma=\frac{5\sqrt{3}}{5}=\sqrt{3}$

$\Rightarrow \gamma=60°=\frac{\pi}{3}$

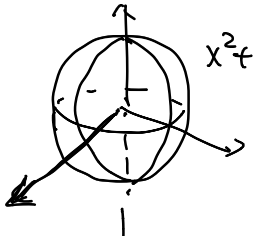
$$\iiint \underline{dV} = \int_{\underline{0}}^{2\pi} \underline{\int\int \rho^2\sin\phi\, d\rho\, \underline{d\phi}\, d\theta}.$$

$$\| $$

$$\int_0^{\frac{\pi}{3}}\int_0^{\frac{5}{\sin\phi}} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{10}$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{\frac{5}{\sin\phi}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{10} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

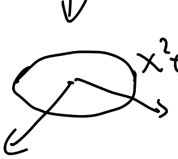
Ex:

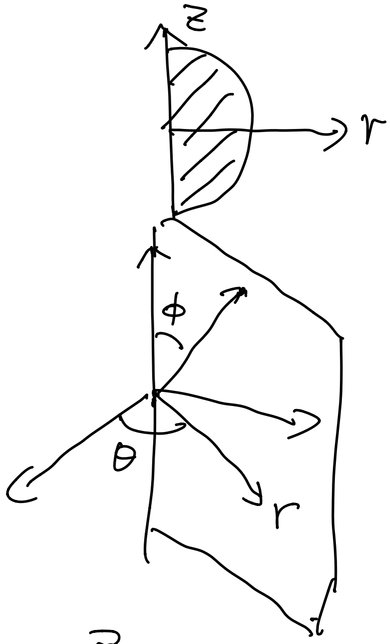


$x^2+y^2+z^2=R^2 \iff \rho=R.$

$$\iiint dV=\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

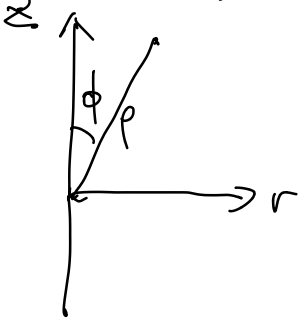
$$=\int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\Big|_0^R \sin\phi\, d\phi\, d\theta$$





$$= \frac{R^3}{3}\int_0^{2\pi} \underset{\substack{\| \\ -\cos\phi\big|_0^{\pi} = 2}}{\left(\int_0^{\pi} \sin\phi\, d\phi\right)} d\theta$$

$$= \frac{R^3}{3}\cdot 2\cdot \int_0^{2\pi} d\theta = \underset{\substack{\| \\ \text{Vol}\left(\substack{\text{ball of} \\ \text{radius } R}\right)}}{\boxed{\frac{4\pi}{3}R^3}}$$



$$\begin{cases} z = \rho\cdot\cos\phi \\ r = \sqrt{x^2+y^2} = \rho\cdot\sin\phi. \end{cases}$$

$\{z>0, r=0\} = \{\phi = 0\}$

$\{z<0, r=0\} = \{\phi = \pi\}.$

xy-plane $= \{z=0\} = \{\phi = \frac{\pi}{2}\}.$