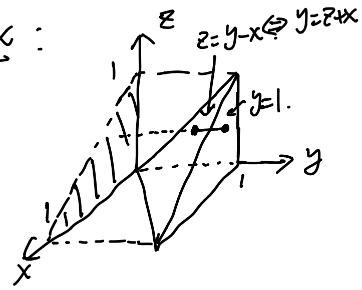
Ex:



tetrahedron bounded by $x=0$, $z=0$, $y=1$, $z=y-x$.

$dx\,dy\,dz$ $\quad dx\,dz\,dy$

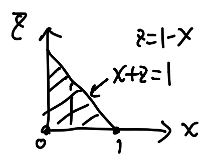
$dy\,dx\,dz$ $\quad dy\,dz\,dx$

$dz\,dx\,dy$ $\quad dz\,dy\,dx$

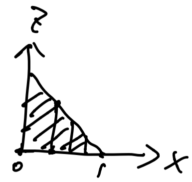
$$\iiint_D dV = \int_0^1 \int_0^{1-x} \int_{z+x}^{1} dy\,dz\,dx$$

1. Project to the xz-plane to get shadow

2. y-limits: $\int_{z+x}^{1} dy$

3. 

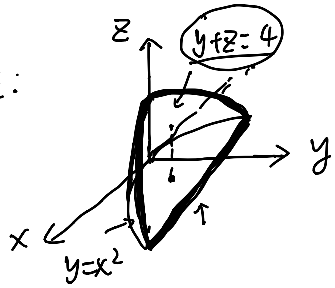
z-limit: $\int_0^{1-x} dz$.

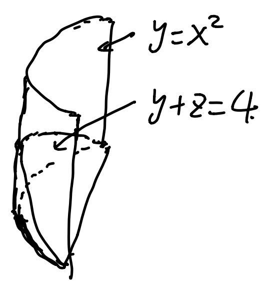


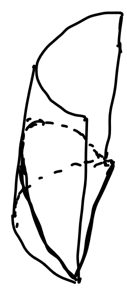
$$\int_0^1 \int_0^{1-x} dz\,dx$$

4. x-limit $\int_0^1 dx$.

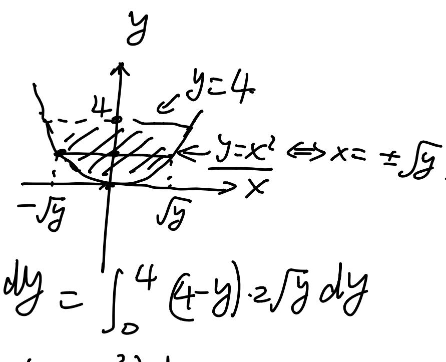
Ex:







$y+z=4$, $y=x^2$, $z=0$.



$$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{4-y} dz\,dx\,dy$$

z: limit: $\int_0^{4-y}$

x limit: $\int_{-\sqrt{y}}^{\sqrt{y}}$

y-limit: $\int_0^4$

$$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} (4-y)\,dx\,dy = \int_0^4 (4-y)\cdot 2\sqrt{y}\,dy$$

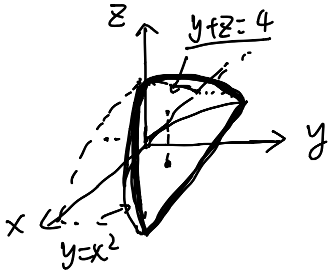
$$= \int_0^4 \left(8\cdot y^{\frac{1}{2}} - 2\cdot y^{\frac{3}{2}}\right) dy$$

$$= \left(8\cdot\frac{2}{3}\cdot y^{\frac{3}{2}} - 2\cdot\frac{2}{5} y^{\frac{5}{2}}\right)\Big|_0^4 = \boxed{\frac{16}{3}\times 8 - \frac{4}{5}\times 2^5}$$

$$\int \int \int dy\, dx\, dz$$

$$\boxed{\begin{array}{c} y+z \le 4 \\ y \ge x^2 \\ z \ge 0 \end{array}}$$

1. shadow:



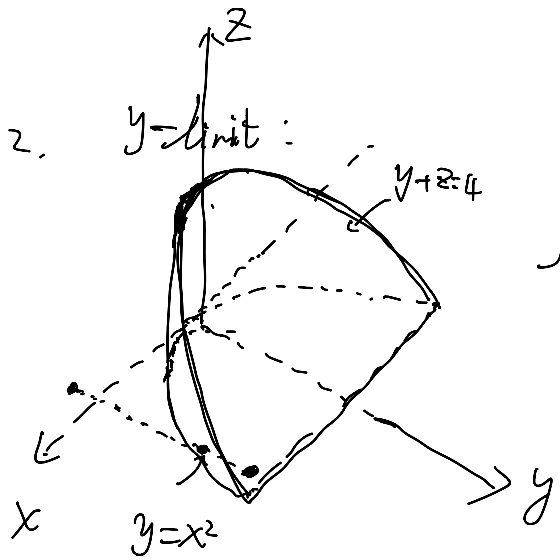
$y+z=4$

$y=x^2$

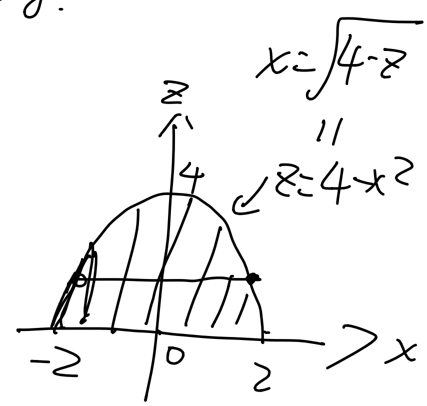
$y=4-z=x^2$

$\boxed{z=4-x^2}$

2. y-limit:



$$\int_{x^2}^{4-z} dy.$$



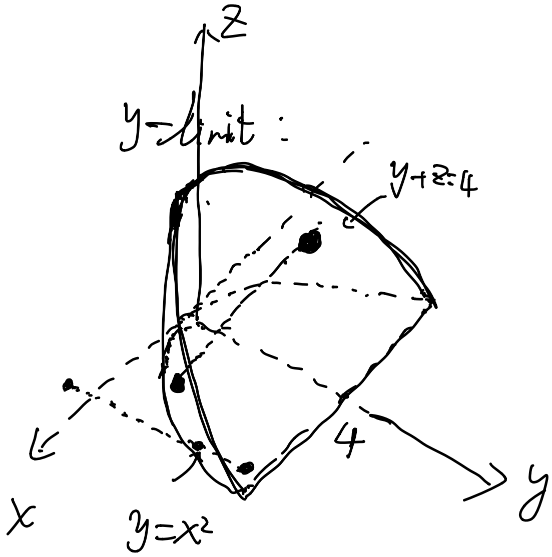
3. x-limit: $\int_{-\sqrt{4-z}}^{\sqrt{4-z}} dx$

4. z-limit: $\int_0^4 dz.$

$$\boxed{\int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} \int_{x^2}^{4-z} dy\,dx\,dz} = ?\ \frac{8\times 32}{15}$$

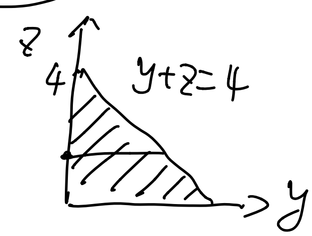
$$\int \int \int \quad dx\,dy\,dz$$

y-limit :



$y+z=4$

$y=x^2$



x-limit : $\int_{-\sqrt{y}}^{\sqrt{y}} dx$

y-limit: $\int_0^{4-z}$

z-limit: $\int_0^4$

$$\boxed{\int_0^4 \int_0^{4-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz}$$

$$\int_0^4 \int_0^{4-z} \underbrace{2\sqrt{y}}_{2\cdot y^{\frac12}}\, dy\cdot dz = \int_0^4 2\cdot\frac{2}{3}\cdot(4-z)^{\frac32}\,dz.$$

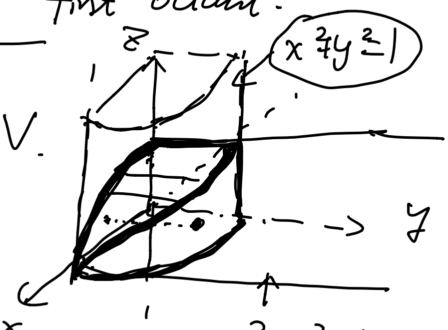
$$= \int_0^4 \frac{4}{3}\cdot t^{\frac32}\,dt = \frac43\times\frac25\cdot t^{\frac52}\Big|_0^4$$

$$= \frac{8}{15}\times 2^5 = \frac{8\times32}{15}$$

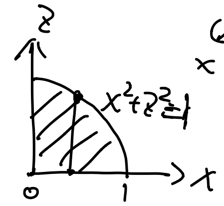
$$\boxed{\frac{16}{3}\times 8 - \frac45\times 2^5} = \frac{16\times8\times5 - 4\times32\times3}{15} = \frac{4\times32\times(5-3)}{15} = \frac{8\times32}{15}$$

Ex: $x^2+y^2=1$, $x^2+z^2=1$ in first octant.

$$\int\int\int dy\,dz\,dx = \iiint_D dV.$$



1. Shadow to the xz-plane



2. y-limit: $\int_0^{\sqrt{1-x^2}} dy$

3. z-limit: $\int_0^{\sqrt{1-x^2}} dz.$

4. x-limit: $\int_0^1 dx$

$$\boxed{\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dy\,dz\,dx}$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dz\,dx$$

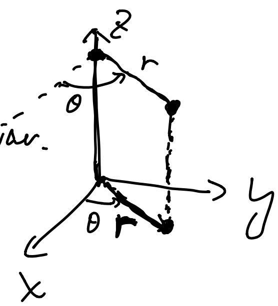
$$= \int_0^1 (1-x^2)\,dx = \left(x-\frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3}.$$

Calculate Triple integral in Cylindrical coordinates.

(r, θ, z)

$\uparrow$

polar coord. of xy-var.



$$\iiint f(x,y,z)\,dV$$

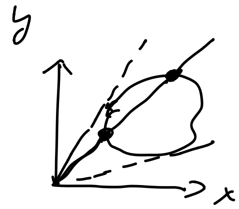
$$\| $$

$$\iint \int f(x,y,z)\,dz\,dx\,dy \to r\,dr\,d\theta.$$

$$\|$$

$$\iint \int f(x,y,z)\,dz\; r\,dr\,d\theta$$

1. shadow to the xy-plane



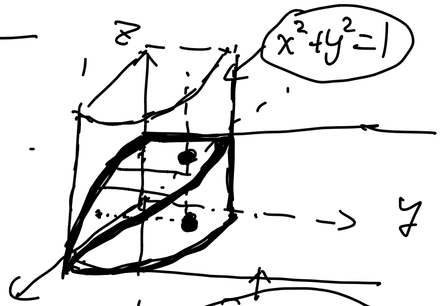
2. z-limits: $\displaystyle\int_{f_1(r,\theta)}^{f_2(r,\theta)} dz$

3. r-limits: $\displaystyle\int_{r_1(\theta)}^{r_2(\theta)} dr.$

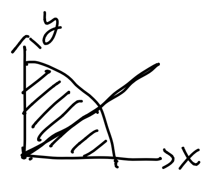
4. θ-limits: $\displaystyle\int_{\theta_{min}}^{\theta_{max}} d\theta$

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \end{cases}$$

Ex:



$x^2+z^2=1. \to z = \sqrt{1-x^2}$



- z-limit: $\displaystyle\int_0^{\sqrt{1-x^2}} = \int_0^{\sqrt{1-r^2\cos^2\theta}}$
- r-limit: $\displaystyle\int_0^1$
- θ-limit: $\displaystyle\int_0^{\frac{\pi}{2}}$

$$\iiint dV = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2\cos^2\theta}} \underline{dz}\, r\, dr \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2\cos^2\theta}\, \underbrace{r dr}_{= d\left(\frac{r^2}{2}\right)}\, d\theta$$

$u = 1-r^2\cos^2\theta$

$du = -2 \cdot r\cos^2\theta\, dr$

$r dr = -\dfrac{du}{2\cos^2\theta}$

$$= \int_0^{\frac{\pi}{2}} \int_1^{1-\cos^2\theta} \underbrace{\sqrt{u}}_{u^{\frac{1}{2}}} \cdot \left(\ominus \frac{du}{2\cos^2\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos^2\theta} \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{\frac{3}{2}} \Big|_{\underbrace{1-\cos^2\theta}_{\sin^2\theta}}^{1} \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\theta} \cdot \frac{1}{3} \cdot \left(1 - \sin^3\theta\right) d\theta .$$

$d\theta = \dfrac{dt}{\sqrt{1-t^2}}$

$\theta = \sin^{-1} t$

$$\frac{1}{3} \frac{1-\sin^3\theta}{1-\sin^2\theta} = \frac{1}{3} \cdot \frac{1-t^3}{1-t^2} = \frac{1}{3} \cdot \frac{1+t+t^2}{1+t} \cdot \frac{dt}{\sqrt{1-t^2}}$$

$\sin\theta = t$

$$= \int_0^1 \frac{1}{3} \cdot \frac{1+t+t^2}{(1+t)} \cdot \frac{1}{\sqrt{1-t^2}}\, dt = \frac{2}{3}$$

$$\int \frac{d\theta}{\cos^2\theta} = \tan\theta, \qquad \int \frac{\sin^3\theta}{\cos^2\theta} d\theta = \int \frac{(1-\cos^2\theta)\cdot \sin\theta\, d\theta}{\cos^2\theta} = -\int \left(\frac{1}{\cos^2\theta} - 1\right) \cdot d(\cos\theta)$$

$$= \frac{1}{\cos\theta} + \cos\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{3} \cdot \frac{1-\sin^3\theta}{\cos^2\theta} d\theta = \frac{1}{3} \cdot \left[\tan\theta - \frac{1}{\cos\theta} - \cos\theta\right]\Big|_0^{\frac{\pi}{2}} = \frac{1}{3} \left[\left(\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta - 1}{\cos\theta} - 0\right) - (0-2)\right]$$

$$= \frac{1}{3} \cdot \left[\left(\lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{-\sin\theta}\right) + 2\right] = \boxed{\frac{2}{3}} \checkmark$$