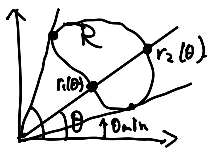
Double integrals in polar coordinates.

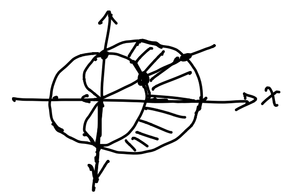


$$\iint_R f(x,y)\,\underset{dxdy}{\underset{\parallel}{dA}} = \int_{\theta_{min}}^{\theta_{max}}\int_{r_1(\theta)}^{r_2(\theta)} \underset{f(r\cos\theta,\, r\sin\theta)}{\uparrow}\; r\,dr\,d\theta$$

$$\boxed{\iint_R dA = \text{Area}(R)}$$

<u>Ex</u>: Area inside <u>cardioid</u> $r = 1 + \cos\theta$ and $\underbrace{}_{\text{outside}}$ <u>circle</u> $r = 1$.

$$1+\cos\theta \overset{}{=} 2\cos^2\frac{\theta}{2}$$



$$\iint_R dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$\cos\frac{\theta}{2} = 0 \Leftrightarrow \frac{\theta}{2} = \frac{\pi}{2}, -\frac{\pi}{2} \Updownarrow \theta = \pi, -\pi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\Big|_1^{1+\cos\theta}\right) d\theta$$

$$1 \le r = 1+\cos\theta \le 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \left((1+\cos\theta)^2 - 1\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\underset{\frac{1+\cos(2\theta)}{2}}{\underset{\parallel}{\cos^2\theta}} + 2\cos\theta\right) d\theta = \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4} + 2\sin\theta\right)\Big|_0^{\frac{\pi}{2}} = \left(\frac{\pi}{4} + 2\right) - 0 = \frac{\pi}{4} + 2$$

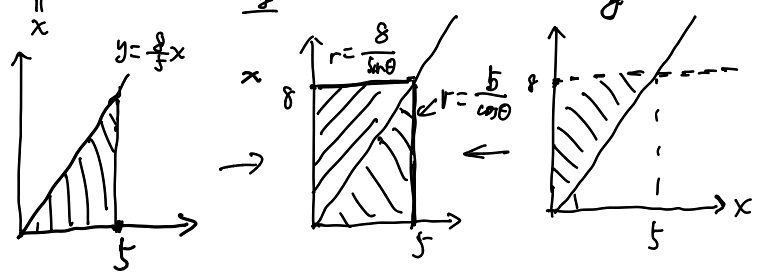
<u>Ex</u>:

$$\underline{\int_0^{\tan^{-1}\left(\frac{8}{5}\right)}\int_0^{\overset{\frac{5}{\cos\theta}}{\overset{\parallel}{5\sec\theta}}} r^7\,dr\,d\theta} + \int_{\tan^{-1}\left(\frac{8}{5}\right)}^{\frac{\pi}{2}}\int_0^{\overset{\frac{8}{\sin\theta}}{\overset{\parallel}{8\csc\theta}}} r^7\,dr\,d\theta.$$

$$\underline{0 \le r \le \frac{5}{\cos\theta},\quad 0 \le \theta \le \tan^{-1}\left(\frac{8}{5}\right).} \qquad 0 \le r \le \frac{8}{\sin\theta},\quad \underline{\tan^{-1}\left(\frac{8}{5}\right) \le \theta \le \frac{\pi}{2}}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$0 \le \underset{x}{\underset{\parallel}{r\cdot\cos\theta}} \le 5,\quad 0 \le \underset{y}{\underset{\parallel}{\tan\theta}} \le \frac{8}{5}.\ \text{or } y \le \frac{8}{5}x \qquad 0 \le \underset{y}{\underset{\parallel}{r\cdot\sin\theta}} \le 8,$$

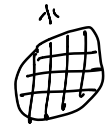


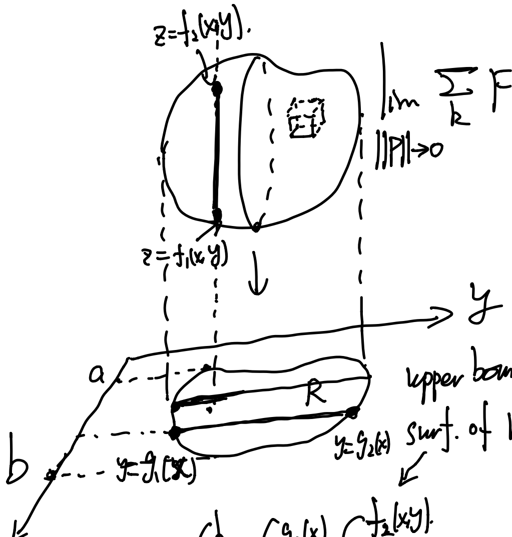
Triple integrals

$$\iiint_D F(x,y,z)\,dV = \lim_{\|P\|\to 0}\sum_k F(x_k,y_k,z_k)\cdot \underbrace{\Delta x_k\cdot\Delta y_k\,\Delta z_k}_{dx\,dy\,dz}$$

volume element

$$\iint_R f(x,y)\,dA = \int_c^d\int_{a(y)}^{b(y)} f(x,y)\,dx\,dy = \int_a^b\int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$





upper bounding surf. of D

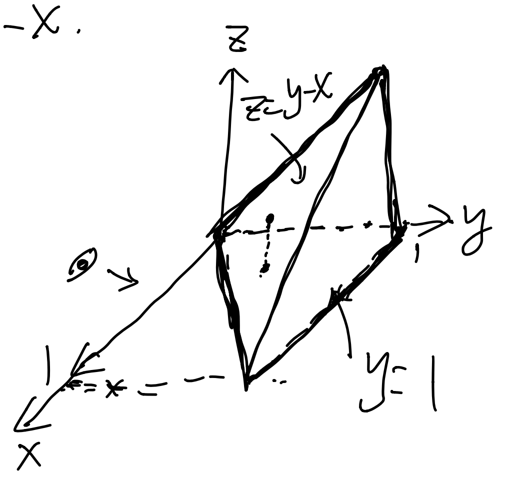
$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

lower bounding surf. of D

dz dy dx	dz dx dy
dy dz dx	dy dx dz
dx dy dz	dx dz dy.

Ex: Find the volume of a tetrahedron bounded by the planes

$x=0,\quad z=0,\quad y=1,\quad z=y-x.$

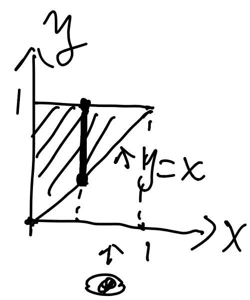


$z=0,\ y=1$

$y=1\quad z=y-x=1-x$

$$\iiint dV = \frac{1}{3}\cdot\text{base}\cdot\text{height} = \frac{1}{3}\times\left(\frac{1}{2}\right)\times 1 = \frac{1}{6}$$

$$\int_0^1 \int_x^1 \int_0^{y-x} 1 \; dz\, dy\, dx$$



$$\int_0^1 \int_x^1 \left(z \Big|_0^{y-x} \right) dy\, dx$$

$$\parallel$$

$$\int_0^1 \int_x^1 (y-x)\, dy\, dx = \int_0^1 \left(\frac{y^2}{2} - xy \right) \Bigg|_x^1 dx$$

$$\parallel$$

$$\int_0^1 \left[\left(\frac{1}{2} - x \right) - \left(\frac{x^2}{2} - x^2 \right) \right] dx$$

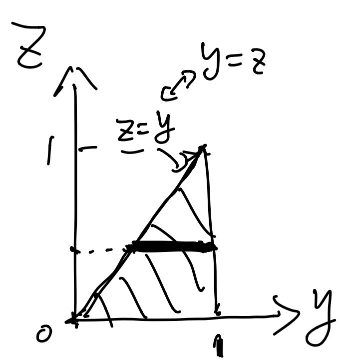
$$\parallel$$

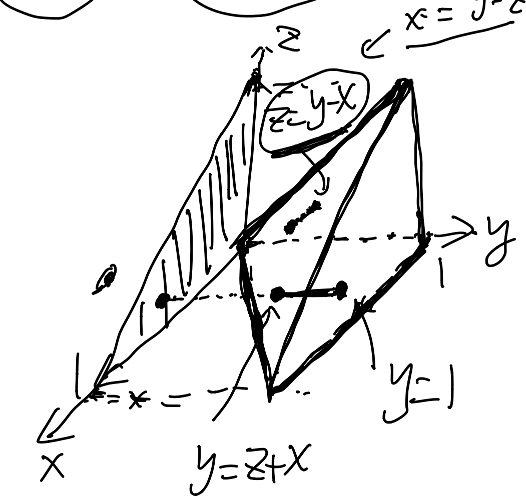
$$\frac{1}{6}$$

$$\parallel$$

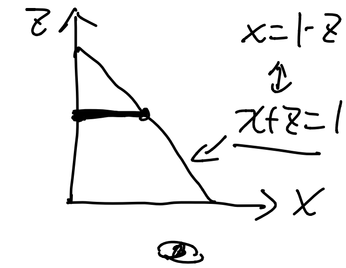
$$\frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \left(\frac{1}{2}x - \frac{1}{2}x^2 + \frac{x^3}{6} \right) \Bigg|_0^1 = \int_0^1 \left[\frac{1}{2} - x + \frac{x^2}{2} \right] dx$$

$$\int_0^1 \int_z^1 \int_0^{y-z} dx\, dy\, dz$$





$$\int_0^1 \int_0^{1-z} \int_{z+x}^{1} \quad dy\, dx\, dz$$



$$= \int_0^1 \int_0^{1-z} \left(1-(z+x)\right) dx\, dz$$

$$= \int_0^1 \left(x - z\cdot x - \frac{x^2}{2}\right)\Big|_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z)^2 - \frac{(1-z)^2}{2}\right] dz = \int_0^1 \frac{1}{2}(1-z)^2\, dz$$

$= \quad u = 1-z$

$$\int_0^1 \frac{1}{2} u^2\, du$$

$$= \frac{u^3}{6}\Big|_0^1 = \frac{1}{6}. \quad ✓.$$