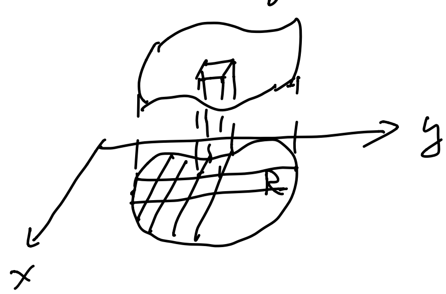
Double integral

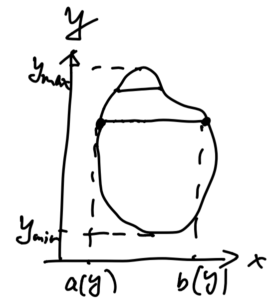



- $\iint_R f(x,y)\,dA$ = volume of the space between the graph and the $R$.

$$= \lim_{\|P\|\to 0} \sum f(x_k,y_k)\,\Delta x_k \Delta y_k.$$

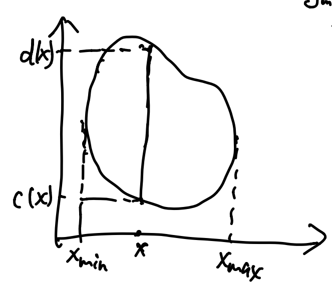
- $\iint_R 1\cdot dA = \iint_R dA$ area of the region $A$.

$$= \lim_{\|P\|\to 0} \sum_k \Delta x_k \Delta y_k$$

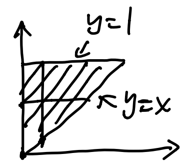



$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_{y_{min}}^{y_{max}} \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

$$= \int_{x_{min}}^{x_{max}} \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$




Ex:




$x \geq 0$
$y \leq 1$
$y \geq x$.

$$\iint_R xy^2\,dx\,dy = \int_0^1 \int_0^y xy^2\,dx\,dy = \int_0^1 y^2 \cdot \frac{x^2}{2}\Big]_0^y dy$$
$$= \int_0^1 \frac{y^4}{2}\,dy = \frac{y^5}{10}\Big|_0^1 = \frac{1}{10}.$$

$$\iint_R xy^2\,dx\,dy = \int_0^1 \int_x^1 xy^2\,dy\,dx = \int_0^1 x\cdot \frac{y^3}{3}\Big|_x^1 dx$$
$$= \int_0^1 \left(\frac{x}{3} - \frac{x^4}{3}\right)dx = \left(\frac{x^2}{6} - \frac{x^5}{15}\right)\Big|_0^1 = \frac{1}{6} - \frac{1}{15} = \frac{5-2}{30} = \frac{3}{30} = \frac{1}{10}$$

Ex: $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 5y \, dx \, dy$

$0 \le y \le 1.$

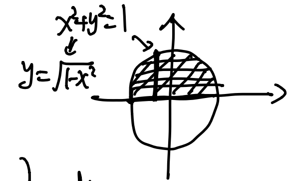
$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \iff |x| \le \sqrt{1-y^2}$

$x^2 \le 1-y^2$

$\boxed{x^2+y^2 \le 1.}$

$\parallel$

$\boxed{\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} 5y \, dy \, dx}$




$\parallel$

$\int_{-1}^{1} \frac{5}{2} y^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{5}{2} \left((1-x^2)-0\right) dx$

$= 2 \cdot \int_0^1 \frac{5}{2} (1-x^2) \, dx = 5 \cdot \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 5 \times \left(1-\frac{1}{3}\right) = \frac{10}{3}.$

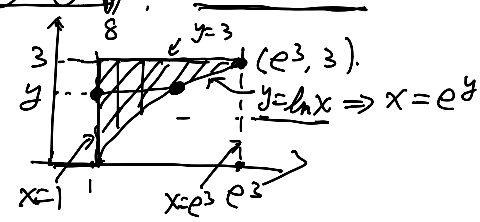
$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 5y \, dx \, dy = \int_0^1 5y \cdot 2\sqrt{1-y^2} \, dy = \int_0^1 5 \cdot (1-y^2)^{\frac{1}{2}} \, \underbrace{d(1-y^2)}_{=-2y\,dy} \cdot (-1)$

$\overset{u=1-y^2}{=} -5 \cdot \int_1^0 u^{\frac{1}{2}} \, du = 5 \cdot \int_0^1 u^{\frac{1}{2}} \, du$

$= 5 \cdot \frac{1}{\frac{1}{2}+1} \cdot u^{\frac{3}{2}} \Big|_0^1 = 5 \times \frac{2}{3} \times 1 = \frac{10}{3}$

Ex: $\int_1^{e^3} \int_{\ln x}^{3} xy \, dy \, dx$ ← vertical

$1 \le x \le e^3$, $\ln x \le y \le 3$




$\parallel$

$\boxed{\int_0^3 \int_1^{e^y} xy \, dx \, dy}$

$\parallel$

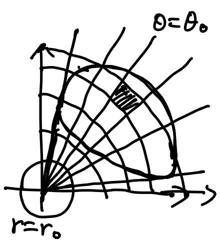
$\int_0^3 y \cdot \frac{x^2}{2} \Big|_1^{e^y} dy = \frac{1}{2} \int_0^3 \left(y e^{2y} - y\right) dy$ use integration by parts

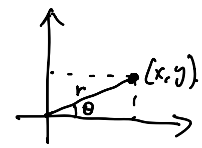
$\int_1^{e^3} \int_{\ln x}^{3} xy \, dy \, dx = \int_1^{e^3} x \cdot \frac{y^2}{2} \Big|_{\ln x}^{3} dx = \frac{1}{2} \int_1^{e^3} x \cdot \left(9 - (\ln x)^2\right) dx.$

Double integrals in polar coordinates.

$\blacksquare \; \Delta x_k \Delta y_k$

$$\iint_R f\, dA = \lim_{\|P\| \to 0} \sum \underline{f(P_k)} \Delta A_k$$

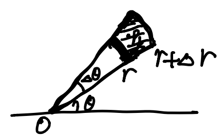



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\|$$

$$\boxed{\iint_R f(r,\theta)\, r\,dr\,d\theta}$$




$$\frac{1}{2}(r+\Delta r)^2 \Delta\theta - \frac{1}{2} r^2 \Delta\theta.$$

$$\| $$

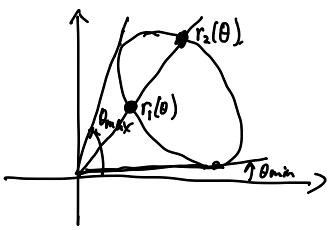
$$\frac{1}{2}(\cancel{r^2} + 2r\Delta r + \Delta r^2)\Delta\theta - \cancel{\frac{1}{2} r^2 \Delta\theta}$$

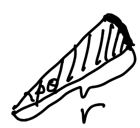
$$\|$$

$$r\Delta r\Delta\theta + (\Delta r)^2 \Delta\theta$$

$$\downarrow$$

$$r\,dr\,d\theta$$



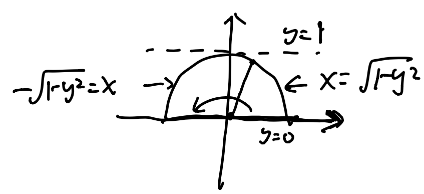

$$\frac{1}{2} r^2 \Delta\theta$$

$$\int_{\theta_{min}}^{\theta_{max}} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\,dr\,\underline{d\theta}$$

Ex:

$$\underline{\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 5\underline{y}\, dx\, dy}$$

$0 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$




$$\|$$

$$\int_0^{\pi} \int_0^1 \underline{5 \cdot r \sin\theta}\, r\,dr\, \underline{d\theta}$$

$$\|$$

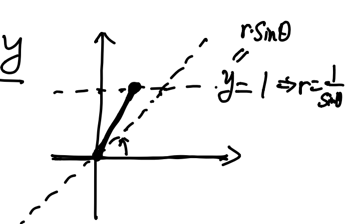
$$\int_0^{\pi} 5 \cdot \sin\theta \cdot \frac{r^3}{3}\Big|_0^1 d\theta = \int_0^{\pi} 5 \cdot \underline{\sin\theta} \cdot \frac{1}{3} d\theta = \frac{5}{3} (-\cos\theta)\Big|_0^{\pi}$$

$$= \frac{5}{3} \cdot (-\cos\pi + \cos 0) = \frac{10}{3}$$

Ex:

$$\int_{\underline{0}}^1 \int_0^y xy^2\, dx\, dy$$

$\underline{0 \le y \le 1}$, $\underline{0 \le x \le y}$




$$\|$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta}} \underline{r} \cos\theta (\underline{r} \sin\theta)^2\, \underline{r}\, dr\, d\theta$$

$$\|$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \int_0^{\frac{1}{\sin\theta}} \cdot r^4 \cdot \underline{\cos\theta \cdot \sin^2\theta}\, dr\, d\theta$$

$$\| $$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\cdot\sin^2\theta\cdot\left(\frac{1}{5}r^5\Big|_0^{\frac{1}{\sin\theta}}\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\cdot\cancel{\sin^2\theta}\cdot\frac{1}{5}\cdot\frac{1}{\underset{\sin^3\theta}{\cancel{\sin^5\theta}}}\,d\theta$$

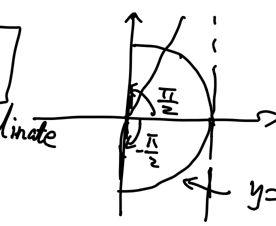
$$= \frac{1}{5}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta\, d\theta}{\sin^3\theta}\ \left(\cos\theta\,d\theta = d\sin\theta\right) \xrightarrow{u=\sin\theta} \frac{1}{5}\int_{\frac{1}{\sqrt2}}^{1}\underbrace{\frac{du}{u^3}}_{=u^{-3}du} = \frac{1}{5}\cdot\frac{1}{-3+1}\cdot u^{-2}\Big|_{\frac{1}{\sqrt2}}^{1}$$

$$= -\frac{1}{10}\left(\frac{1}{1^2}-\frac{1}{\left(\frac{1}{\sqrt2}\right)^2}\right) = -\frac{1}{10}(1-2) = \frac{1}{10}$$

**Ex:** $\displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-\overbrace{(x^2+y^2)}^{r^2}}\,dy\,dx$ $\quad 0\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$

$\boxed{\int e^{-y^2}dy\ ?}$

don't know how to integrate in rectangular coordinate




$y=-\sqrt{4-x^2}$

$\Updownarrow$

$x^2+y^2=4$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}\,\underbrace{r\,dr}_{=\frac{1}{2}dr^2}\,d\theta$$

$$\overset{u=r^2}{=}\ \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^4 e^{-u}du\right)d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-e^{-u}\Big|_0^4\right)d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-e^{-4}+1\right)d\theta = \frac{1-e^{-4}}{\cancel{2}}\cdot\cancel{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{2}(1-e^{-4})$$