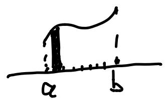
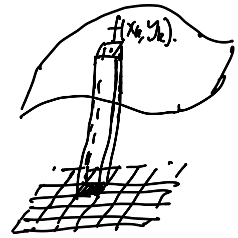
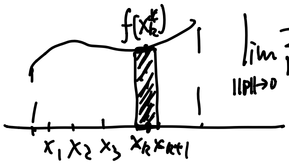
15.1 Double integrals and iterated integrals



$\int_a^b f(x)\,dx$ = area

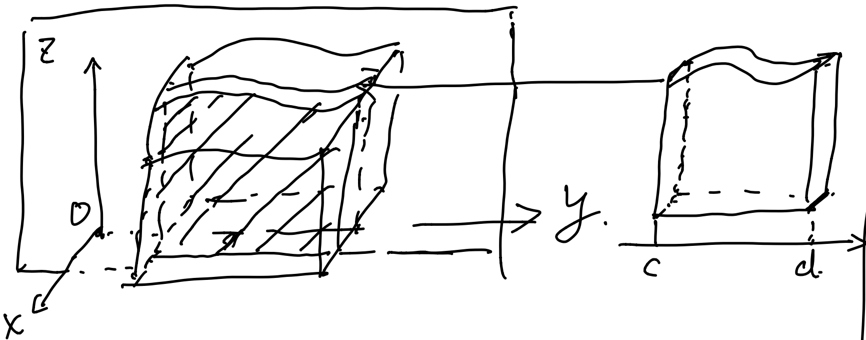


Double integral



$$\lim_{\|P\|\to 0}\sum_{k=1}^{N} f(x_k^*)\cdot \underbrace{\Delta x_k}_{x_{k+1}-x_k} = \int_a^b f(x)\,dx.$$

$$\lim_{\|P\|\to 0}\sum_{k=1}^{N} f(x_k, y_k)\cdot \Delta A_k = \iint_R f(x,y)\,dA$$



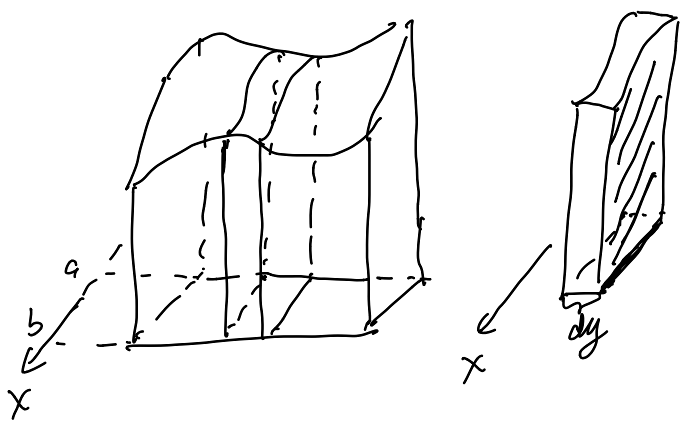
$$= \iint_R f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

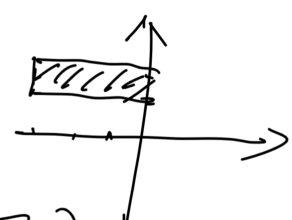
iterated integral

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's Thm



$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$



Ex: $R: -3 \le x \le 0,\ 1 \le y \le 2$

$$\boxed{f(x,y) = x + x^2 y}$$

$$\iint_R f(x,y)\underbrace{dA}_{dx\,dy} = \int_1^2 \boxed{\int_{-3}^{0}(x+x^2y)\,dx}\,dy$$

$$= \int_1^2 \left(-\frac{9}{2} + 9y\right) dy.$$

(Newton-Leibniz: $\int_a^b f(x)dx = F(b)-F(a)$
where $F' = f$.)

$$\int_{-3}^{0}(x+x^2y)\,dx = \left(\frac{1}{2}x^2 + \frac{1}{3}x^3y\right)\Big|_{-3}^{0}$$

$$= 0 - \left(\frac{1}{2}\cdot(-3)^2 + \frac{1}{3}\cdot(-3)^3\cdot y\right)$$

$$= -\frac{9}{2} + 9\cdot y.$$

$$\int_1^2 \left(-\frac{9}{2} + 9y\right) dy = \left(-\frac{9}{2}y + \frac{9}{2}y^2\right)\Big|_1^2$$

$$= \left(-\frac{9}{2}2 + \frac{9}{2}2^2\right) - \left(-\frac{9}{2}1 + \frac{9}{2}1^2\right)$$

$$= (-9+18) - (0) = 9.$$

$$\iint_R f(x,y)\,dA = \int_{-3}^{0} \int_{1}^{2} (x+x^2y)\,dy\,dx$$

$$\int_1^2 (x+x^2y)\,dy = \left[x\cdot y + x^2\cdot\frac{1}{2}y^2\right]_1^2$$

$$= \left(x\cdot 2 + x^2\cdot\frac{1}{2}2^2\right) - \left(x\cdot 1 + x^2\cdot\frac{1}{2}\cdot 1^2\right)$$

$$= (2x+2x^2) - \left(x+\frac{1}{2}x^2\right)$$

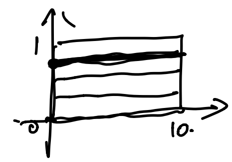
$$= x + \frac{3}{2}x^2$$

$$\int_{-3}^{0} \left(x+\frac{3}{2}x^2\right)dx = \left[\frac{1}{2}x^2 + \frac{1}{2}x^3\right]_{-3}^{0}$$

$$= 0 - \left(\frac{1}{2}\cdot(-3)^2 + \frac{1}{2}(-3)^3\right)$$

$$= -\frac{9}{2} + \frac{27}{2} = \frac{18}{2} = \underline{9}$$

Ex: R: $0 \le x \le 10$, $0 \le y \le 1$. $f(x,y) = \frac{yx^4}{y^2+1}$

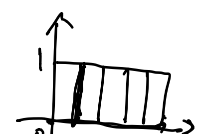


$$\iint_R f(x,y)\,dx\,dy = \int_0^1 \underline{\int_0^{10} \frac{yx^4}{y^2+1}\,dx}\,dy$$

$$= \int_0^1 \left(\frac{y}{y^2+1}\cdot\frac{1}{5}x^5\Big|_0^{10}\right)dy = \int_0^1 \frac{y}{y^2+1}(2\times 10^4)\cdot dy$$

$$= 10^4\cdot\int_0^1 \frac{2y\,dy}{y^2+1} \quad 2y\,dy = d(y^2+1) \quad \xrightarrow{u=y^2+1} 10^4\cdot\int_1^2 \frac{du}{u} = 10^4\cdot \ln u\Big|_1^2 = \underline{10^4\cdot\ln 2}$$

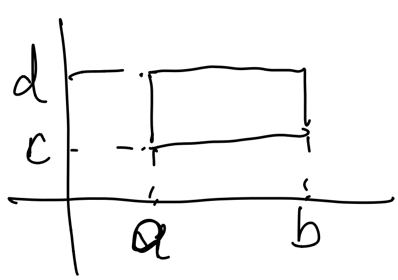
$$= \int_0^{10}\int_0^1 \frac{yx^4}{y^2+1}\,dy\,dx = \int_0^{10} x^4\cdot\int_0^1 \frac{d(y^2+1)}{y^2+1}\cdot\frac{1}{2}\,dx$$



$$= \int_0^{10} \frac{1}{2}x^4\left(\int_1^2 \frac{du}{u}\right)dx = \int_0^{10}\frac{1}{2}x^4\cdot\left(\ln u\Big|_1^2\right)\cdot dx$$

$$= \int_0^{10}\frac{1}{2}x^4\ln 2\cdot dx = \frac{1}{2}\ln 2\cdot\frac{1}{5}x^5\Big|_0^{10} = \frac{1}{2}\ln 2\cdot\frac{1}{5}\cdot 10^5 = \underline{10^4\cdot\ln 2}$$

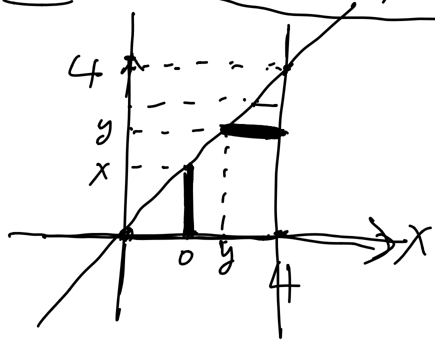
15.2: More general regions



$$\iint_R dA = \int_c^d\int_a^b dx\,dy \quad \text{(horizontal slice)}$$

$$= \int_a^b\int_c^d dy\,dx \quad \text{vertical slice}$$

Ex: $0 \le x \le 4,\quad 0 \le y \le x.$



$$\iint_R f\,dA = \int_0^4 \int_0^x f\,dy\,dx$$

$$= \int_0^4 \int_y^4 f\,dx\,dy$$

$$f(x,y) = x \cdot y^2.$$

$$\int_0^4 \int_0^x (xy^2)\,dy\,dx = \int_0^4 \left(x \cdot \frac{1}{3} y^3 \Big|_0^x\right) dx$$

$$= \int_0^4 \left(x \cdot \frac{1}{3} \cdot x^3 - \underbrace{x \cdot \frac{1}{3} \cdot 0^3}_{0}\right) dx$$

$$= \int_0^4 \frac{1}{3} \cdot x^4\,dx = \frac{1}{15} x^5 \Big|_0^4 = \frac{4^5}{15} = \frac{4^5}{15}$$

$$\int_0^4 \int_y^4 (x \cdot y^2)\,dx\,dy = \int_0^4 \left(y^2 \cdot \frac{1}{2} x^2 \Big|_y^4\right) dy$$

$$= \int_0^4 \left(y^2 \cdot \frac{1}{2} \cdot 4^2 - y^2 \cdot \frac{1}{2} \cdot y^2\right) dy$$

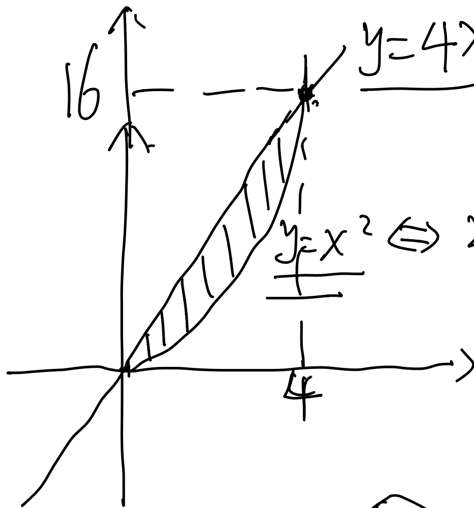
$$= \int_0^4 \left(8y^2 - \frac{1}{2} y^4\right) dy$$

$$= \left(\frac{8}{3} y^3 - \frac{1}{10} y^5\right)\Big|_0^4$$

$$= \frac{8}{3} 4^3 - \frac{1}{10} 4^5 = 4^4 \left(\frac{2}{3} - \frac{2}{5}\right) = 4^4 \cdot \frac{10-6}{15} = \frac{4^5}{15}.$$

Ex:

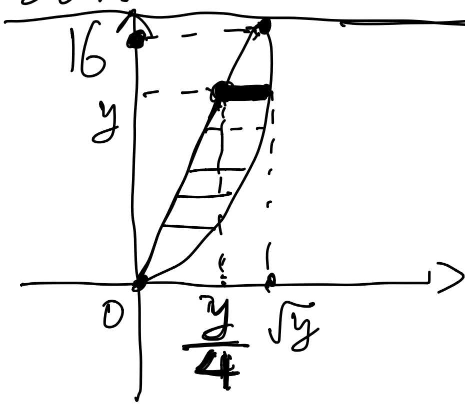
$$\begin{cases} y=4x \\ y=x^2 \end{cases}$$
$$\Downarrow$$
$$x^2=4x$$
$$\Downarrow$$
$$x=4 \text{ or } x=0$$

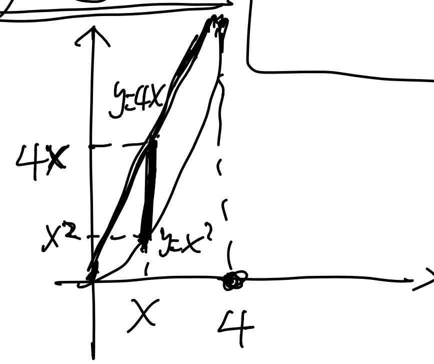


$$\iint_R f(x,y)\,dA$$

$$f=1 \Rightarrow \underbrace{\iint_R 1\,dA}_{=\ \text{area of } R}$$

$$\iint_R 1\cdot dA = \int_0^{16}\int_{\frac{y}{4}}^{\sqrt{y}} dx\,dy$$





$$\iint_R 1\cdot dA = \int_0^4\int_{x^2}^{4x} dy\,dx$$
$$= \int_0^4 (4x-x^2)\,dx = \left(2x^2-\frac{1}{3}x^3\right)\Big|_0^4$$
$$= 2\times 4^2 - \frac{1}{3}4^3 = 4^2\cdot\left(2-\frac{4}{3}\right) = \frac{4^2\times 2}{3} = \frac{32}{3}$$

$$\int_0^{16}\int_{\frac{y}{4}}^{\sqrt{y}} dx\,dy = \int_0^{16}\left(\overset{y^{\frac{1}{2}}}{\sqrt{y}} - \frac{y}{4}\right)dy$$

$$= \left(\frac{1}{\frac{1}{2}+1}\cdot y^{\frac{1}{2}+1} - \frac{1}{8}y^2\right)\Big|_0^{16}$$

$$= \left(\frac{2}{3}\cdot y^{\frac{3}{2}} - \frac{1}{8}y^2\right)\Big|_0^{16}$$

$$= \frac{2}{3}\cdot 16^{\frac{3}{2}} - \frac{1}{8}16^2$$

$$16^{\frac{3}{2}} = (16^{\frac{1}{2}})^3 = 4^3$$

$$= \frac{2}{3}\times 4^3 - 2\times 4^2$$

$$= 2\times 4^2\cdot\left(\frac{4}{3}-1\right) = 32\cdot\frac{1}{3}.$$