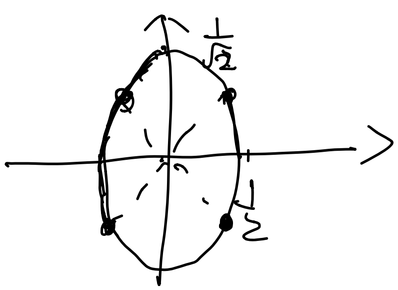
14.8 Lagrange Multipliers: Find constraint max/min

Ex: Find the points on the ellipse $4x^2+2y^2=1$ where $f(x,y)=xy$ has its extreme value.



$$\frac{x^2}{\frac{1}{4}}+\frac{y^2}{\frac{1}{2}}=1 \qquad a^2=\frac{1}{4} \qquad \frac{1}{2}=b^2$$

Method 1:

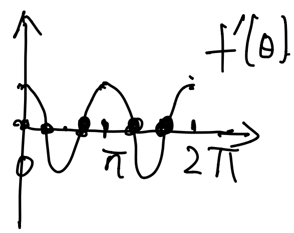
$$\begin{cases} x=\frac{1}{2}\cos\theta = a\cdot\cos\theta \\ y=\frac{1}{\sqrt{2}}\sin\theta = b\cdot\sin\theta \end{cases}$$

$0\le\theta\le 2\pi$

$$f(x,y)=x\cdot y=\frac{1}{2}\cos\theta\cdot\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2\sqrt{2}}\sin\theta\cos\theta=\frac{1}{4\sqrt{2}}\sin(2\theta)$$

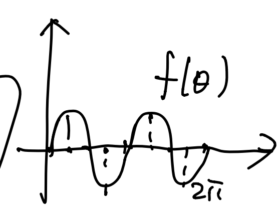
$$f(x(\theta),y(\theta))=f(\theta)=\frac{1}{4\sqrt{2}}\sin(2\theta)$$

$$f'(\theta)=\frac{1}{4\sqrt{2}}\cos(2\theta)\cdot 2=\frac{1}{2\sqrt{2}}\cos(2\theta)=0$$



$$\Rightarrow \theta=\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$$

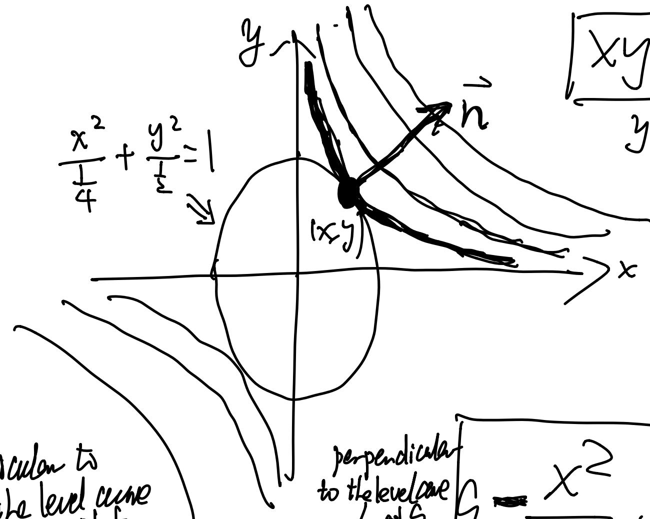
$$f(\theta)=\frac{1}{4\sqrt{2}},\ -\frac{1}{4\sqrt{2}},\ \frac{1}{4\sqrt{2}},\ -\frac{1}{4\sqrt{2}}$$



$$f(0)=0=f(2\pi).$$

Method 2:

$f(x,y) = x \cdot y$



$\boxed{xy = C}$

$y = \frac{c}{x}$

perpendicular to the level curve of f

perpendicular to the level curve of g

$$\boxed{g = \frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{2}} - 1}$$

$$\boxed{\vec{n} \parallel \nabla f, \quad \vec{n} \parallel \nabla g}$$

multiplier

$$\boxed{\nabla f \parallel \nabla g} \Rightarrow \boxed{\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}}$$

constraint

$f = x \cdot y, \quad \nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle$

$g = \frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{2}} - 1 = 4x^2 + 2y^2 - 1, \quad \nabla g = \langle 8x, 4y \rangle.$

$$\begin{cases} \langle y, x \rangle = \lambda \langle 8x, 4y \rangle \Leftrightarrow \begin{cases} y = \lambda \cdot 8x = \lambda \cdot 8 \cdot \lambda \cdot 4y = 32\lambda^2 \cdot y \\ x = \lambda \cdot 4y \end{cases} \\ 4x^2 + 2y^2 - 1 = 0 \end{cases}$$

$y \cdot (32\lambda^2 - 1) = 0$

$\Downarrow$

- $y = 0 \Rightarrow x = 0$ ✗
- $32\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm \frac{1}{\sqrt{32}}$

$\lambda^2 = \frac{1}{32}$

$\lambda = \pm \frac{1}{4\sqrt{2}}$

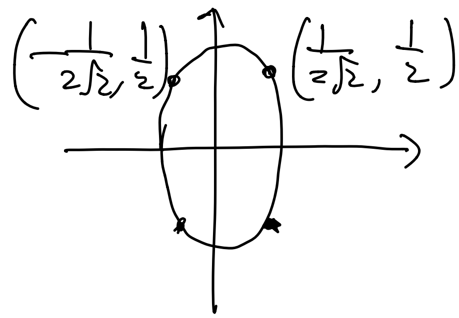
$4 \cdot x^2 + 2 \cdot \lambda^2 \cdot 64 \cdot x^2 = 1$

$\left(4 + 2 \cdot \frac{1}{\cancel{32}} \times \cancel{64}^{\,2}\right) x^2 = 1$

$8 \cdot x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{8}} = \pm \frac{1}{2\sqrt{2}}$

x	y	λ
$\frac{1}{2\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{4\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{4\sqrt{2}}$
$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{4\sqrt{2}}$
$-\frac{1}{2\sqrt{2}}$	$+\frac{1}{2}$	$-\frac{1}{4\sqrt{2}}$

$$y = 8\lambda \cdot x = 8 \times \frac{1}{4\sqrt{2}} \times \frac{1}{2\sqrt{2}} = \frac{1}{2}$$



$$\begin{cases} x = \frac{1}{2}\cos\theta \\ y = \frac{1}{\sqrt{2}}\sin\theta \end{cases}$$

$\theta = \frac{\pi}{4}$, $x = \frac{1}{2}\cos\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$

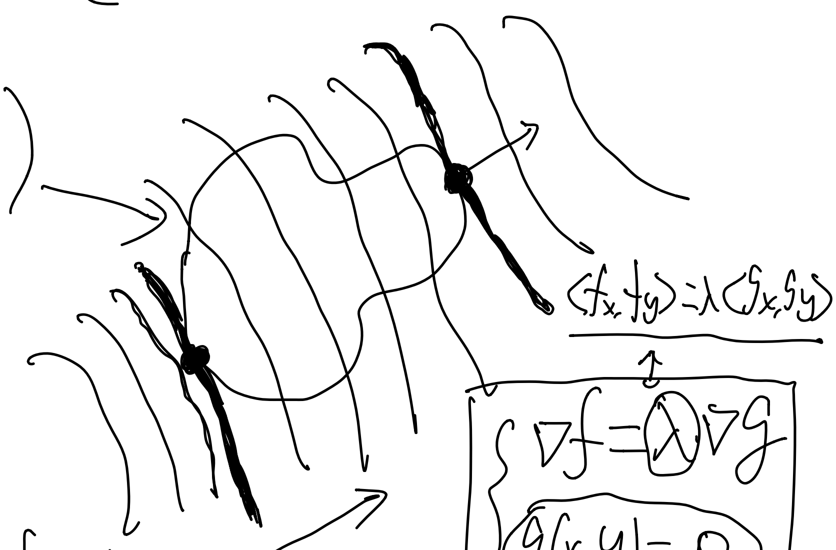
$y = \frac{1}{\sqrt{2}}\cdot\sin\theta = \frac{1}{2}$

$\left(\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right)$

$(g(x,y) = 0)$

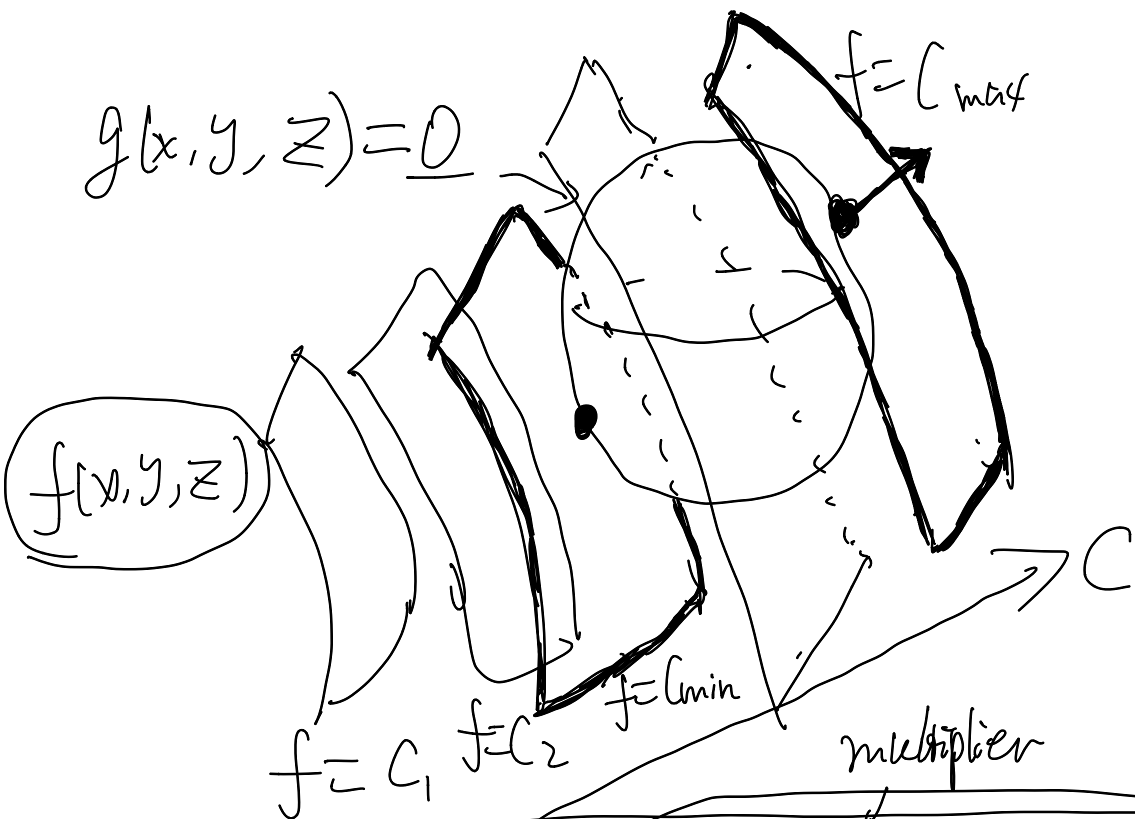
maximize/minimize

$f(x,y)$



$\langle f_x, f_y\rangle = \lambda\langle g_x, g_y\rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$



multiplier

$$\begin{cases} \nabla f=\lambda\cdot\nabla g \iff \langle f_x, f_y, f_z\rangle=\lambda\langle g_x, g_y, g_z\rangle \\ g(x,y,z)=0 \end{cases}$$

Ex: Find max/min of $f(x,y,z)=3x-7y+8z$ on the sphere $x^2+y^2+z^2=122$

$\nabla f=\langle f_x, f_y, f_z\rangle=\langle 3,-7,8\rangle$

$\nabla g=\langle 2x, 2y, 2z\rangle$.

$g=x^2+y^2+z^2-122$

$$\begin{cases} 3 = \lambda \cdot 2x \to x = \frac{3}{2\lambda} \\ -7 = \lambda \cdot 2y \to y = -\frac{7}{2\lambda} \\ 8 = \lambda \cdot 2z \to z = \frac{8}{2\lambda} = \frac{4}{\lambda} \\ x^2+y^2+z^2 = 122 \end{cases}$$

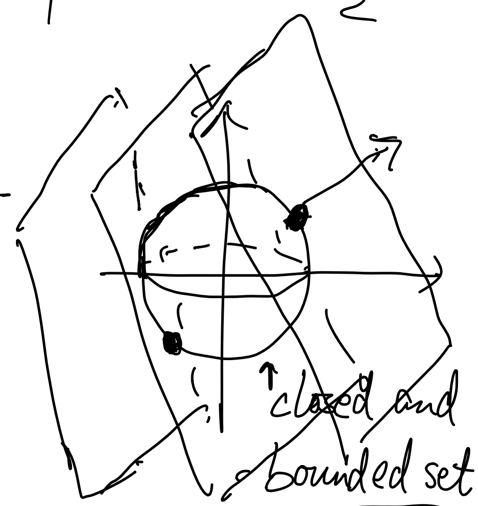
$\nabla f = \lambda \nabla g$

$$\frac{9}{4\lambda^2} + \frac{49}{4\lambda^2} + \frac{64}{4\lambda^2} = 122$$

$$\begin{array}{r} 58 \\ 64 \\ \hline 122 \end{array}$$

$$\underbrace{\frac{1}{4\lambda^2}\cdot \cancel{122}}_{1} = \cancel{122} \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$$

x	y	z	λ
(3	−7	8)	$\frac{1}{2}$
(−3	7	−8)	$-\frac{1}{2}$



Fact: A continuous function on a closed and bounded set always obtains the absolute maximum and absolute minimum.

$f = 3x - 7y + 8z$

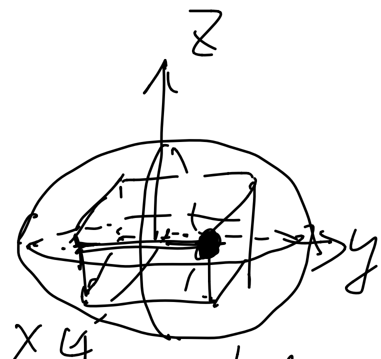
$f(3,-7,8) = 3\times 3 - 7\times(-7) + 8\times 8 = 9 + 49 + 64$

$= 122 > 0$

abs. max

$f(-3,7,-8) = \underline{-122}$ abs. min.

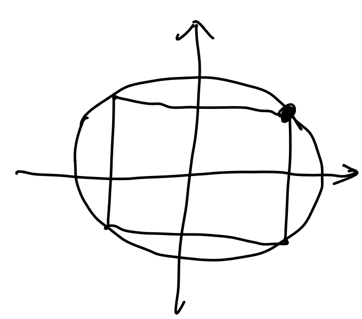
Ex: $\dfrac{x^2}{4} + \dfrac{y^2}{2} + z^2 = 1.$



Find the dimension of the closed rectangular box with the maximum volume that can be inscribed in the ellipsoid and whose faces are parallel to the coordinate planes

Find (x,y,z) s.t. $x>0,\ y>0,\ z>0$

s.t. $\boxed{g(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{2} + z^2 - 1 = 0.}$



and $\boxed{f(x,y,z) = (2x)\cdot(2y)\cdot(2z) = \underline{8xyz}}$ obtains the max.

$$\underline{\nabla f} = \langle 8yz,\ 8xz,\ 8xy \rangle = \langle 8\times\frac{\sqrt{2}}{3},\ 8\cdot\frac{2}{3},\ \frac{16\sqrt{2}}{3} \rangle = \frac{8\sqrt{2}}{\sqrt{3}}\nabla g$$

$$\underline{\nabla g} = \langle \frac{x}{2},\ y,\ 2z \rangle = \underline{\langle \frac{1}{\sqrt{3}},\ \frac{\sqrt{2}}{\sqrt{3}},\ \frac{2}{\sqrt{3}} \rangle}$$

$$a^2 = \frac{1}{2}$$

$$\nabla f = \lambda\cdot\nabla g \qquad \begin{cases} 8yz = \lambda\cdot\frac{x}{2} \\ 8\cdot xz = \lambda\cdot y \\ 8xy = \lambda(2z) \\ \boxed{\frac{x^2}{4} + \frac{y^2}{2} + z^2 = 1} \end{cases}$$

$$\to \frac{y}{x} = \frac{1}{2}\cdot\frac{x}{y} \qquad \frac{y}{x} = a \qquad a = \frac{1}{2a} \Rightarrow a = \frac{1}{\sqrt{2}} = \frac{y}{x}$$

$$\underbrace{\frac{z}{x}}_{b} = \frac{1}{4}\cdot\frac{x}{z} \qquad b = \frac{1}{4b} \Rightarrow b = \frac{1}{2} = \frac{z}{x}$$

$$y = \frac{1}{\sqrt{2}}\cdot x, \quad z = \frac{1}{2}\cdot x.$$

$$\frac{x^2}{4} + \frac{\frac{x^2}{2}}{2} + \frac{x^2}{4} = \frac{3}{4}\cdot x^2 = 1 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}.$$

$$y = \frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}, \quad z = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \quad = \quad \frac{4}{3}\cdot\frac{1}{4} + \frac{2}{3\times 2} + \frac{1}{3} \ \checkmark$$

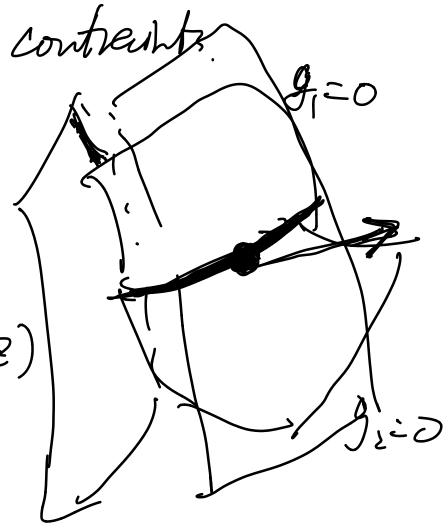
$$\lambda = \frac{16yz}{x} = \frac{16\times\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} = \frac{8\sqrt{2}}{\sqrt{3}}$$

f obtains the maximum at point $\left(\frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = P_{max}$

$$\boxed{f(P_{max}) = 8\times\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{16\sqrt{2}}{3\sqrt{3}}}$$

(length $\frac{4}{\sqrt{3}}$, width $\frac{2\sqrt{2}}{\sqrt{3}}$, height $= \frac{2}{\sqrt{3}}$)

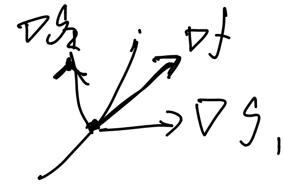
· Lagrangian multiplier with 2 contraints.



$\underline{g_1(x,y,z)=0}$, $\underline{g_2(x,y,z)=0}$.

Find extremal value of $f(x,y,z)$ under these 2 constraints.

$$\nabla f \perp \left(\underbrace{\{g_1=0\}\cap\{g_2=0\}}_{C}\right)$$



$\underline{\nabla g_1} \perp C$, $\underline{\nabla g_2 \perp C}$

$\Downarrow$

multipliers

$$\begin{cases} \underline{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2} \leftarrow 3 \text{ equations.} \\ g_1(x,y,z)=0 \\ g_2(x,y,z)=0. \end{cases}$$

3+2 equations

3+2 variables

$\rightsquigarrow$ solve to get (x,y,z,λ,μ).

<u>Ex</u>: $\{z^2=x^2+y^2\}\cap\{x+y+z=1\}=C$

Find a point on C that closest to the origin.

$g_1 = x^2+y^2-z^2$, $\quad g_2 = x+y+z-1$

$f(x,y,z) = \underline{x^2+y^2+z^2 = (\text{distance to the origin})^2}$

$\nabla g_1 = \langle 2x, 2y, -2z \rangle$, $\quad \nabla g_2 = \langle 1, 1, 1 \rangle$.

$\nabla f = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} 2x = \lambda \cdot 2x + \mu \cdot 1 \\ 2y = \lambda \cdot 2y + \mu \cdot 1 \\ 2z = \lambda(-2z) + \mu \cdot 1 \\ x^2+y^2-z^2=0 \\ x+y+z-1=0 \end{cases}$$

$2(x-y) = 2\lambda \cdot (x-y)$.

$(x-y)\cdot(\lambda-1)=0$

$\Rightarrow \cdot\, x=y$

$\cdot\, \cancel{\lambda = 1}$

$\cdot\ \lambda = 1.\ \mu = 0. \Rightarrow 2z = -2z \Rightarrow \underline{z=0} \Rightarrow x^2+y^2=0$

$\Rightarrow x = y = 0$

$\overline{x+y+z=0 \neq 1}$

$$\begin{cases} 2x = \lambda \cdot 2x + \mu \\ 2z = \lambda \cdot (-2z) + \mu \\ 2x^2 - z^2 = 0 \\ 2x+z-1=0 \end{cases}$$

$x - z = \lambda \cdot (x+z)$.

$z = \pm\sqrt{2}\, x$.

$2x \pm \sqrt{2}x = 1 \Rightarrow x = \frac{1}{2+\sqrt{2}}$ or $\frac{1}{2-\sqrt{2}}$

	x	y	z	λ	μ
P_{min}	$\frac{1}{2+\sqrt{2}}$	$\frac{1}{2+\sqrt{2}}$	$\frac{\sqrt{2}}{2+\sqrt{2}}$		
P_{max}	$\frac{1}{2-\sqrt{2}}$	$\frac{1}{2-\sqrt{2}}$	$-\frac{\sqrt{2}}{2-\sqrt{2}}$		

$$f(P_{min}) = \left(\frac{1}{2+\sqrt{2}}\right)^2 + \left(\frac{1}{2+\sqrt{2}}\right)^2 + \left(\frac{\sqrt{2}}{2+\sqrt{2}}\right)^2$$

$P_{min} = \left(\frac{1}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}}, \frac{\sqrt{2}}{2+\sqrt{2}}\right)$ closest pt. on C to the origin.

$$f(P_{max}) = \left(\frac{1}{2-\sqrt{2}}\right)^2 + \left(\frac{1}{2-\sqrt{2}}\right)^2 + \left(\frac{\sqrt{2}}{2-\sqrt{2}}\right)^2$$

$P_{min} = \left(\frac{1}{2-\sqrt{2}}, \frac{1}{2-\sqrt{2}}, -\frac{\sqrt{2}}{2-\sqrt{2}}\right)$ furthest pt. on C to the origin