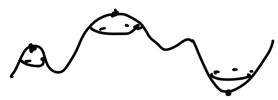
$z=f(x,y)$.



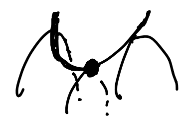
that is an interior

$\boxed{\text{1st derivative test}}$: if f achieves a local maximum or local minimum at (x_0,y_0) and if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist, then $f_x(x_0,y_0)=0=f_y(x_0,y_0)$.

Critical point: (x_0,y_0) is a critical point if $f_x(x_0,y_0)=0=f_y(x_0,y_0)$ or one or both of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ do not exist.

local extremal points are critical points.

critical point may not be local extrema, because it may be a saddle point



Thm: 2nd derivative test for local extreme values.

Assume · f, f_x, f_y, f_{xx}, f_{xy}, f_{yy} all exist and are continuous

symmetric matrix

· $f_x(a,b)=f_y(a,b)=0$. Then look at Hessian matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ \underset{f_{xy}}{\underset{\shortparallel}{f_{yx}}} & f_{yy}\end{pmatrix}$

$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix} = f_{xx}f_{yy}-f_{xy}^2$

i) if $\boxed{f_{xx}<0}$ at (a,b) and $\boxed{f_{xx}f_{yy}-f_{xy}^2>0}$, then f has local maximum at (a,b).

ii) .. $f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$, ... minimum ...

iii) $f_{xx}f_{yy}-f_{xy}^2<0$, ... $\boxed{\text{saddle point}}$ ..

iv) $f_{xx}f_{yy}-f_{xy}^2=0$, then inconclusive.

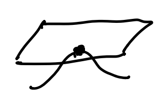
(Compare to $y=f(x)$. · 1st der. test. local extrema satisfies $f'(a)=0$.

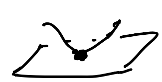
· 2nd der. test. $f''(a)>0 \Rightarrow$ local minimum

$f''(a)<0 \Rightarrow$ local maximum

$f''(a)=0 \Rightarrow$ inconclusive.

$\boxed{f(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\text{error}}$)





$$f(x,y)=f(a,b)+\underbrace{f_x(a,b)}_{0}\cdot(x-x_0)+\underbrace{f_y(a,b)}_{0}\cdot(y-y_0)+\boxed{\frac{1}{2}\left(f_{xx}(a,b)\cdot(x-a)^2+2f_{xy}(a,b)\cdot(x-a)(y-b)+f_{yy}(a,b)(y-b)^2\right)}$$

$+$ error.

$$f(x,y)=f(a,b)+\boxed{\frac{1}{2}(x-a \quad y-b)\boxed{\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{pmatrix}}_{(a,b)}\begin{pmatrix} x-a \\ y-b\end{pmatrix}}+\cancel{\text{error}}$$

Ex: $f(x,y) = 2x^2+3xy+4y^2-6x+7y$. Find critical points and classify them.

$f_x = 4x+3y-6$, $f_y = 3x+8y+7$

$$\begin{cases} 4x+3y=6 \\ 3x+8y=-7 \end{cases} \longrightarrow x = \frac{\begin{vmatrix} 6 & 3 \\ -7 & 8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & 8 \end{vmatrix}} = \frac{48+21}{32-9} = \frac{69}{23} = 3.$$

$4\times3+3\times(-2)=6$ ✓

$3\times3+8\times(-2)=9-16=-7$

$$y = \frac{\begin{vmatrix} 4 & 6 \\ 3 & -7 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & 8 \end{vmatrix}} = \frac{-28-18}{23} = -\frac{46}{23} = -2.$$

critical point: $(3, 2)$.

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$$

$f_{xy} = f_{yx}$

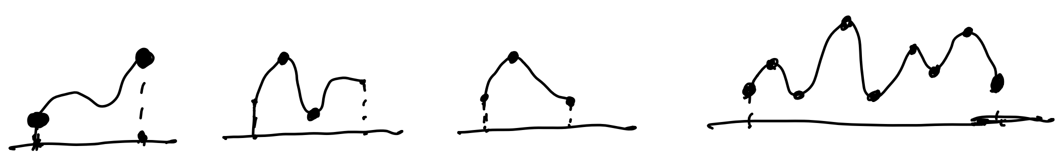
$f_{xx} = 4 > 0$.

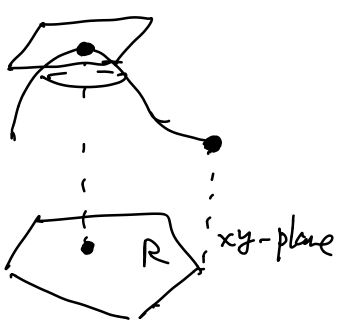
$f_{xx}f_{yy} - f_{xy}^2 = 4\times8-3\times3 = 32-9 = 23 > 0$.

$\Longrightarrow$ f has a local minimum at point $(3,-2)$.

- Absolute maxima and Absolute minima

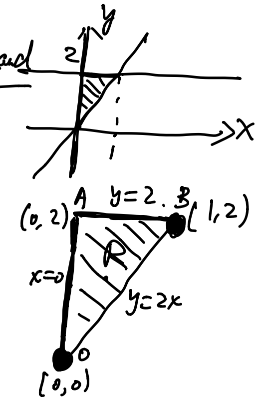
Fact: If $f(x,y)$ is continously defined on a bounded and closed set (R) in the plane then $f(x,y)$ obtains both absolute maxima and absolute minimum over R





Ex: Find absolute max./min. of function $f(x,y) = 2x^2 - 4x + y^2 - 4y + 6$ on the closed triangle bounded by the lines $x=0$, $y=2$, $y=2x$.

step1: List interior points where f may have local extrema and evaluate f at these points.

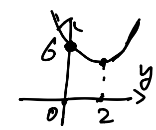


$f_x = 4x - 4 = 0$, $f_y = 2y - 4 = 0$

$\Rightarrow$ critical point $(1, 2)$. over xy-plane

there is no interior critical point.

step 2: List the boundary points of R where f has local maximum/minimum and evaluate.



$\overline{OA}$: $x=0$, $0 \le y \le 2$. $f(0,y) = y^2 - 4y + 6 = (y-2)^2 + 2$.

$f(0,y)$: minimum at $y=2$ $(0,2)$. $f(0,2) = 2$

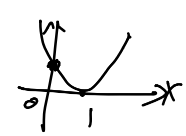
maximum at $y=0$. $(0,0)$ $f(0,0) = 6$.

$\overline{AB}$: $y=2$, $0 \le x \le 1$. $f(x,2) = 2x^2 - 4x + \underbrace{2^2 - 4 \times 2 + 6}_{4-8+6} = 2x^2 - 4x + 2$

$= 2(x^2 - 2x + 1) = 2 \cdot (x-1)^2$.

$f(x,2)$: minimum at $x=1$. $(1,2)$ $f(1,2) = 0$

maximum at $x=0$. $(0,2)$, $f(0,2) = 2$.



$\overline{OB}$: $y = 2x$, $0 \le x \le 1$. $f(x,2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 6$

$= 6x^2 - 12x + 6 = 6 \cdot (x-1)^2$

$f(x,2x)$ minimum at $x=1$, $(1,2)$, $f(1,2) = 0$

maximum at $x=0$, $(0,0)$ $f(0,0) = 6$.

step 3 : abs. min

bdry (x_0, y_0)	$(0,2)$	$(1,2)$
f	2	0

$\Rightarrow$ f obtains absolute minimum at $(1,2)$ with value 0.

abs. max.

	$(0,0)$	$(0,2)$
	6	2

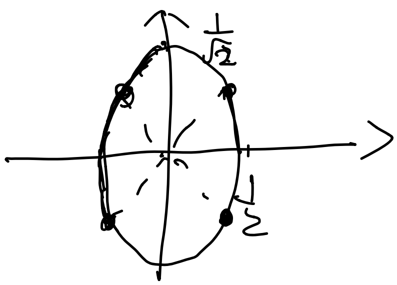
$\Rightarrow$ f obtains absolute maximum at $(0,0)$. with value $f(0,0)=6$.

14.8 Lagrange multiplier

a method for find constrained max/min.

Ex: Find the points on the ellipse $4x^2+2y^2=1$ where $f(x,y)=xy$ has its extreme value.

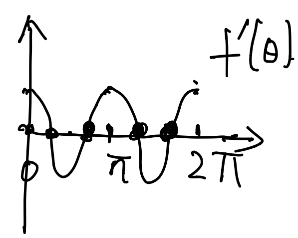
$$\frac{x^2}{\underbrace{\frac{1}{4}}_{a^2=}}+\frac{y^2}{\underbrace{\frac{1}{2}}_{=b^2}}=1.$$



Method 1:
$$\begin{cases} x=\frac{1}{2}\cos\theta=a\cdot\cos\theta \\ y=\frac{1}{\sqrt{2}}\sin\theta=b\cdot\sin\theta \end{cases}$$
$0\leq\theta\leq 2\pi$

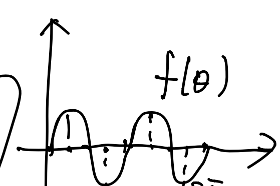
$$f(x,y)=x\cdot y=\frac{1}{2}\cos\theta\cdot\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2\sqrt{2}}\sin\theta\cos\theta=\frac{1}{4\sqrt{2}}\sin(2\theta)$$
$$\overset{\parallel}{f}(x(\theta),y(\theta))=f(\theta)=\frac{1}{4\sqrt{2}}\sin(2\theta).$$

$$f'(\theta)=\frac{1}{4\sqrt{2}}\cos(2\theta)\cdot 2=\underset{\overset{\parallel}{0}}{\frac{1}{2\sqrt{2}}\cos(2\theta)}$$



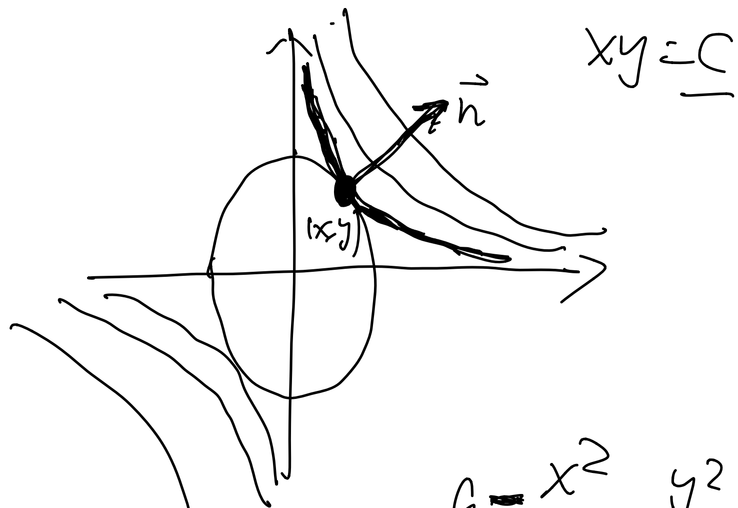
$$\Rightarrow \theta=\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$$

$$\boxed{f(\theta)=\frac{1}{4\sqrt{2}},\ -\frac{1}{4\sqrt{2}},\ \frac{1}{4\sqrt{2}},\ -\frac{1}{4\sqrt{2}}}$$



$$f(0)=0=f(2\pi).$$

Method 2 :



$$\boxed{\vec{n} \parallel \nabla f, \quad \vec{n} \parallel \nabla g}$$

$$g = \frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{2}} - 1$$

multiplier

$$\nabla f \parallel \nabla g \Rightarrow \boxed{\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}}$$