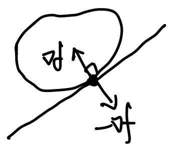
14.6 $\quad z=f(x,y)$. $\quad \nabla f = \langle f_x, f_y \rangle$

$|\vec{u}|=1 \quad \underbrace{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$

rate of change of f in the direction $\vec{u}$.

level curve $f(x,y)=C$. at $P_0(x_0,y_0)$ on the level curve. $\nabla f|_{P_0}$ is normal to the level curve

perpendicular to

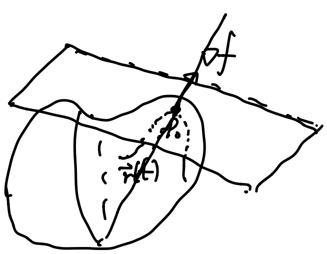


$\frac{\nabla f}{|\nabla f|}$: direction most increasing direction

$-\frac{\nabla f}{|\nabla f|}$: - - - - decreasing direction.

$w=f(x,y,z) \qquad \nabla f = \langle f_x, f_y, f_z \rangle$

level surface $\underline{\{f(x,y,z)=C\}} \ni P_0 \quad \nabla f|_{P_0}$ is perpendicular to the level surface.



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. $\quad \boxed{f(x(t), y(t), z(t)) \equiv C.}$

$(x(t), y(t), z(t)) \in$ level surface $\forall t$.

$\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$

$\boxed{0 = \nabla f \cdot \vec{r}'(t).} \Rightarrow \nabla f \perp \vec{r}'(t)$

$\Rightarrow \nabla f$ perp. to the level surface

$\Rightarrow \underline{\vec{r}'(t)}$ in the plane that is perp. to ∇f.

Equation for the tangent plane:

$$\nabla f|_{P_0} \cdot (\vec{r} - \vec{OP_0}) = 0$$

$$\langle f_x(P_0), f_y(P_0), f_z(P_0) \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0.$$

$$\boxed{f_x(P_0)\cdot(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)\cdot(z-z_0) = 0.}$$

$P_0 = (x_0, y_0, z_0)$

normal line: $\quad \vec{r}(t) = \vec{r}_0 + t \cdot \nabla f|_{P_0}$.

$\sqrt{1-x_0^2+y_0^2} = \sqrt{1-1+2^2} = 2$

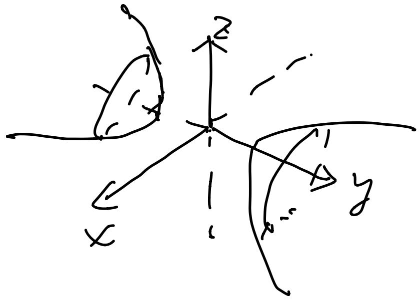
Ex: $\underbrace{\{x^2-y^2+z^2=1\}}_{f(x,y,z)} \ni \underline{(x_0, y_0, z_0)} \qquad$ e.g. $(1,2,2)$

$\underline{\nabla f = \langle 2x, -2y, 2z \rangle.}$

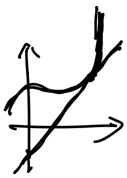
$$\boxed{2x_0\cdot(x-x_0) + (-2y_0)\cdot(y-y_0) + 2z_0(z-z_0) = 0.}$$

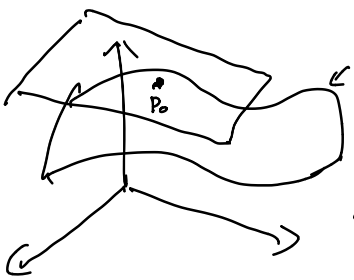
$x^2+z^2=y^2+1 \geq 1$

2-sheeted. hyperboloid



Ex: $\underline{z=f(x,y)}$





$g(x,y,z)=f(x,y)-z$.

$\underbrace{\{f(x,y)-z=0\}}_{\{z=f(x,y)\}} = \underline{\{g(x,y,z)=0\}}$.

$\nabla g = \langle g_x, g_y, g_z \rangle = \langle f_x, f_y, \underline{-1} \rangle$.

$P_0(x_0,y_0,z_0)$.

tangent plane: $f_x(P_0)\cdot(x-x_0)+f_y(P_0)\cdot(y-y_0)-(z-z_0)=0$.

$$\boxed{z=f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+z_0}$$

linearization of $f(x,y)$ at (x_0,y_0).

$z=\overset{f(x,y).}{\overset{\shortparallel}{\sqrt{x-y^2}}}$ $\quad \underline{(12,\ 2,\ 2\sqrt{2})}$.

$f(12,2)=\sqrt{12-2^2}=\sqrt{8}$.

$f_x\Big|_{(12,2)}=\frac{1}{2}\frac{1}{\sqrt{x-y^2}}=\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}=\frac{1}{4\sqrt{2}}$

$f_y\Big|_{(12,2)}=\frac{1}{2}\cdot\frac{-2y}{\sqrt{x-y^2}}=-\frac{y}{\sqrt{x-y^2}}=-\frac{2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}$

tangent plane:

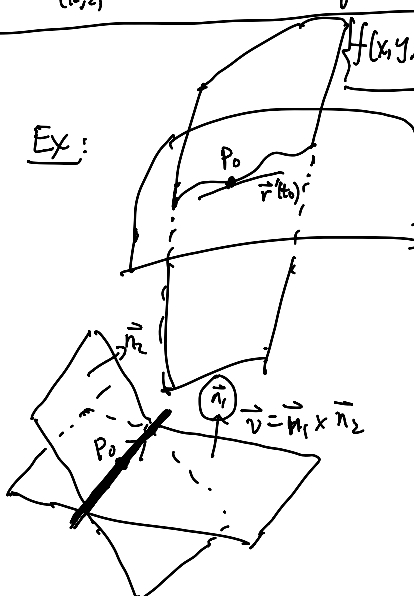
$$\boxed{z=\frac{1}{4\sqrt{2}}\cdot(x-12)+\left(-\frac{1}{\sqrt{2}}\right)(y-2)+2\sqrt{2}}$$

linearization

Ex:

$\{f(x,y,z)=C_1\}=S_1$

$\{g(x,y,z)=C_2\}=S_2$



$\nabla f|_{P_0} \perp S_1, \quad \nabla g|_{P_0} \perp S_2$.

$\Updownarrow$

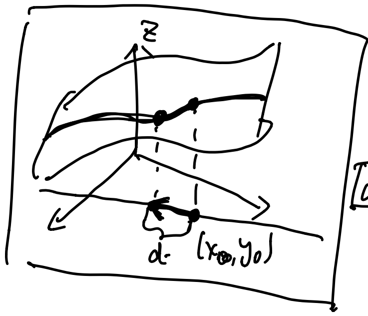
$\nabla f|_{P_0} \perp$ any tangent vector of S_1
$\nabla g|_{P_0} \perp$ . . . . . of S_2

$\Rightarrow$ $\nabla f|_{P_0} \perp \underline{\vec{r}'(t_0)}$ and $\nabla g|_{P_0} \perp \underline{\vec{r}'(t_0)}$

$\Rightarrow$ $\vec{r}'(t_0) = \boxed{\nabla f|_{P_0} \times \nabla g|_{P_0}}$

a vector tangent to the curve of intersection.

- Estimate the change of $f(x,y)$ in a specific direction



Linearization: $\boxed{z-z_0} = f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$

$$\boxed{z-z_0 = \nabla f(x_0,y_0)\bullet\langle x-x_0,\ y-y_0\rangle}$$

differentiability

$$\underbrace{z-z_0}_{=\Delta z} = f_x(x_0,y_0)\cdot\underbrace{(x-x_0)}_{=\Delta x} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{=\Delta y} + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$\varepsilon \to 0$ as $\Delta x, \Delta y \to 0$

Ex: $w = f(x,y,z) = \ln\sqrt{x+y^2+z^3}$. $\quad P_0 = (1,1,1)$

approximate change in the direction of $\underbrace{\vec i + 2\vec j - 3\vec k}_{=\vec v}$ if P_0 moves by $\boxed{\underbrace{ds}_{=d} = 0.1}$ in that direction.

$$\underbrace{\Delta w}_{=df} = \nabla f|_{P_0}\cdot\underbrace{\langle x-x_0,\ y-y_0,\ z-z_0\rangle}_{=d\cdot\frac{\vec v}{|\vec v|}} = \underbrace{\left(\nabla f|_{P_0}\bullet\frac{\vec v}{|\vec v|}\right)}_{=\nabla f|_{P_0}\bullet\vec u}\cdot d.$$

$$\boxed{df = (\nabla f|_{P_0}\bullet\vec u)\cdot ds}$$

$$\boxed{df = (D_{\vec u} f|_{P_0})\cdot ds}$$

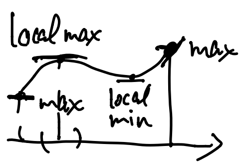
$$\nabla f = \langle \underbrace{f_x}_{=\frac{1}{2}\cdot\frac{1}{x+y^2+z^3}}, f_y, f_z\rangle = \left\langle \frac{1}{2}\cdot\frac{1}{x+y^2+z^3},\ \cancel{\frac{1}{2}}\cdot\frac{\cancel{2}y}{x+y^2+z^3},\ \frac{1}{2}\cdot\frac{3z^2}{x+y^2+z^3}\right\rangle\Big|_{(1,1,1)}$$

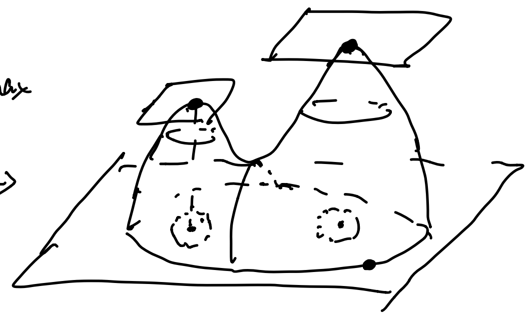
$$= \left\langle \frac{1}{2}\cdot\frac{1}{3},\ \frac{1}{3},\ \frac{1}{2}\cdot\frac{3}{3}\right\rangle = \left\langle \frac{1}{6},\ \frac{1}{3},\ \frac{1}{2}\right\rangle$$

$$\vec u = \frac{\vec i + 2\vec j - 3\vec k}{\sqrt{\underbrace{1^2+2^2+3^2}_{1+4+9}}} = \frac{1}{\sqrt{14}}\langle 1, 2, -3\rangle$$

$$df = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\rangle \bullet \frac{1}{\sqrt{14}}\langle 1, 2, -3\rangle\cdot(0.1) = \frac{1}{\sqrt{14}}\left(\frac{1}{6}+\frac{2}{3}-\frac{3}{2}\right)\cdot(0.1) = \dots$$

14.7: Extremal Values and saddle points





Def: local maximum $f(a,b)$: $f(a,b) \geq f(x,y)$ near (a,b), i.e. $f(a,b) \geq f(x,y)$ for all (x,y) in an open disk centered at (a,b)

Local minimum : . . .

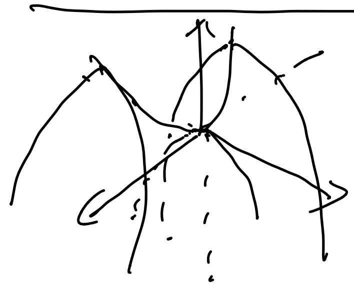
Thm (1st derivative test for local extremal values).
If f has a local maximum or local minimum at an interior point (a,b), and if the 1st partial derivatives exist at (a,b), then $\boxed{f_x(a,b)=0=f_y(a,b)}$.

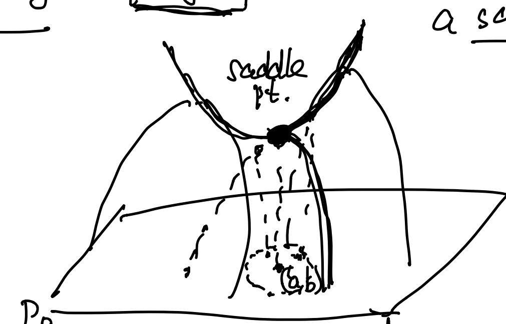
$\left(\begin{array}{l} z = f_x(a,b)\cdot(x-a) + f_y(a,b)\cdot(y-b) + z_0 \\ \boxed{z = z_0} \text{ horizontal tangent plane} \end{array} \right)$

Ex: $z = x^2 - y^2 = f(x,y)$.

$0 = f_x = 2x,\ 0 = f_y = -2y \Rightarrow \boxed{x=y=0}$

a saddle point is a critical pt.



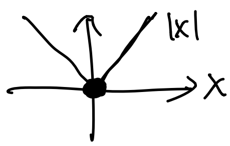


Def: An interior point of the domain of a function $f(x,y)$ is a critical point of f

if either $f_x(P_0) = f_y(P_0) = 0$

or (one or both of $f_x(P_0)$ and $f_y(P_0)$ do not exist.)

Ex:



Ex: $|x|+|y| = f(x,y)$

$\underbrace{f(x,y)}_{|x|+|y|} \geq \underbrace{f(0,0)}_{0}$ is the (absolute) minimum

f_x, f_y do not exist at $\boxed{(0,0)}$

Ex: $f(x,y) = x^2 + xy + 5x + 3y - 2$

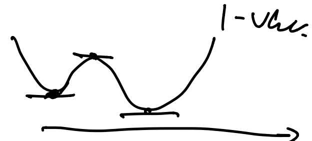
critical pts: $f_x = 2x + y + 5 = 0 \quad y = -5 - 2x = -5 + 6 = 1.$

$f_y = x + 3 = 0 \Rightarrow x = -3$

$\boxed{(-3, 1)}$

local min ?
max ?
saddle ?

1-var.



$f''(x_0) > 0 \to$ local min
$f''(x_0) < 0 \to$ local max
$f''(x_0) = 0$ inconclusive

2nd derivative test for local extrema