Chain rule: $T = T(x, y, z)$ (temperature at $(x,y,z)$.)

$$x = x(r,s),\quad y = y(r,s),\quad z = z(r,s).$$

$$T = T(x(r,s),\ y(r,s),\ z(r,s)) = T(r,s).$$

$$\begin{array}{ccccc} & & x & & \\ r & & y & & T \\ s & & z & & \end{array}$$

$$\frac{\partial T}{\partial r} = \underline{\frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial r}}$$

Ex: $T = x\cos(y - z)$ $\quad x = r+s,\ \underline{y = r-s},\ \underline{z = r\cdot s}.$

$$\frac{\partial T}{\partial r} = \underbrace{\cos(y-z)}_{\frac{\partial T}{\partial x}}\cdot 1 + \underbrace{(-x\cdot\sin(y-z))}_{\frac{\partial T}{\partial y}}\cdot 1 + \left(x\cdot(-\sin(y-z))(-1)\cdot \overset{\frac{\partial z}{\partial r}}{s}\right)$$

$$\frac{\partial T}{\partial z} = x\cdot(-\sin(y-z))\cdot\overset{-1}{\frac{\partial(y-z)}{\partial z}}$$

$$= \cos(y-z) - x\cdot\sin(y-z) + x\cdot\sin(y-z)\cdot s.$$

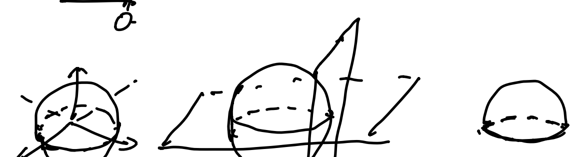
$$= \underline{\cos(r-s-rs) - (r+s)\cdot\sin(r-s-rs) + (r+s)\sin(r-s-rs)\cdot s.} = \frac{\partial T}{\partial r}.$$

Implicit differentiation: $F(x,y,z) = 0 \rightsquigarrow z = z(x,y)$ $\quad \underline{\frac{\partial z}{\partial x}.}$

$$\underline{F(x,y,z(x,y)) \equiv 0.}$$

$$0 = F_x + F_z\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$



Ex: $x^2+y^2+z^2 = 1 \Rightarrow \underline{z = \sqrt{1-x^2-y^2}} \geq 0$

$$F(x,y,z) = x^2+y^2+z^2-1 = 0 \qquad F_x = \frac{\partial F}{\partial x} = 2x \qquad F_z = 2z$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z} = -\frac{x}{z}} \quad z \neq 0$$

$$\begin{array}{l} x \\ y \end{array} - u - z \qquad z = \sqrt{u} \qquad u = 1-x^2-y^2$$

$$\frac{dz}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{u}}\cdot(-2x) = -\frac{x}{\sqrt{1-x^2-y^2}}$$

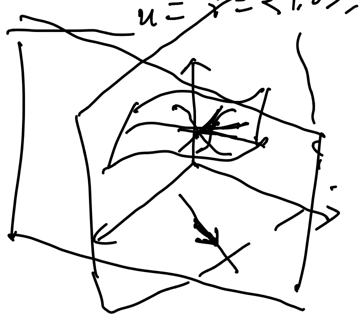
$$\boxed{\frac{\partial z}{\partial x} = -\frac{x}{z}}$$

## 14.5 Directional derivatives and Gradient Vectors

$z=f(x,y)$. $(x_0,y_0)$. choose a direction represented by a unit vector $\vec{u} = \langle u_1,u_2\rangle$, $|\vec{u}|=1$

$$D_{\vec{u}}f\Big|_{P_0} = \frac{d}{ds}f(x_0+s\cdot u_1,\ y_0+su_2)\Big|_{s=0} = \lim_{s\to 0}\frac{f(x_0+su_1,\ y_0+su_2)-f(x_0,y_0)}{s}$$

$\vec{u}=\vec{i}=\langle 1,0\rangle$, $D_{\vec{i}}f\big|_{P_0} = \frac{\partial f}{\partial x}\big|_{P_0}$ $\qquad D_{\vec{j}}f\big|_{P_0} = \frac{\partial f}{\partial y}\big|_{P_0}$.



rate of change of $f$ in the direction $\vec{u}$.

$x = x_0+su_1$, $y = y_0+su_2$

$s \to x, y \to z=f$

$$\frac{df}{ds}\Big|_{P_0} = \frac{\partial f}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds}\Big|_{P_0}$$

$$D_{\vec{u}}f\big|_{P_0} = f_x\cdot u_1 + f_y\cdot u_2.$$

$$= \langle f_x, f_y\rangle \cdot \langle u_1,u_2\rangle$$

$$= \underbrace{\nabla f\big|_{P_0}}_{\text{gradient vector of } f \text{ at } P_0}\cdot\ \vec{u}$$

del

$\nabla f = f_x\vec{i}+f_y\vec{j} = \langle f_x,f_y\rangle$

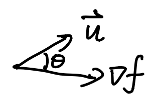
$\nabla f$ = gradf

delf

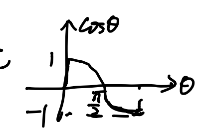
Ex: $f(x,y) = x^2\ln y - \sin(x+2y)$.

$f_x = 2x\cdot\ln y - \cos(x+2y)$, $f_y = x^2\cdot\frac{1}{y} - \cos(x+2y)\cdot 2$.

$\nabla f = \langle f_x, f_y\rangle$



$0\le\theta\le\pi$

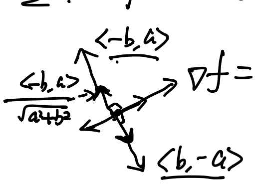


$$D_{\vec{u}}f = \nabla f\cdot\vec{u} = |\nabla f|\cdot\underbrace{|\vec{u}|}_{=1}\cdot\cos\theta = |\nabla f|\cdot\cos\theta$$

$-|\nabla f| \le D_{\vec{u}}f\le|\nabla f|$

1. $D_{\vec{u}}f = |\nabla f|$ if $\theta=0$: $f$ increases most rapidly when $\theta=0$, or when $\vec{u}$ is the direction of $\nabla f$. ($\vec{u} \rightarrow \nabla f$)
2. $D_{\vec{u}}f = -|\nabla f|$ if $\theta=\pi$: $f$ decreases most rapidly when $\theta=\pi$, or when $\vec{u}$ is the opposite direction of $\nabla f$.
3. $D_{\vec{u}}f = 0$ if $\theta=\frac{\pi}{2}$: if $\vec{u}\perp\nabla f$, then $\vec{u}$ is a $\boxed{\text{direction of zero change}}$.

Assume $\nabla f = \langle f_x, f_y\rangle \ne 0$. $\frac{\langle -b,a\rangle}{\sqrt{a^2+b^2}}$




$\langle -b,a\rangle\cdot\langle a,b\rangle = -ba+ab=0$.

$\langle a,b\rangle \to \langle -b, a\rangle$.
$\langle b, -a\rangle$.

Ex: $f(x,y) = x^3 - y^2$ at $(1,2)$.

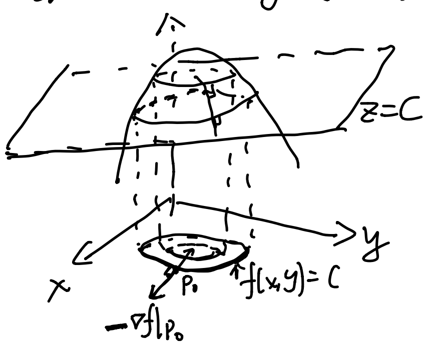
$\nabla f = \langle f_x, f_y\rangle = \langle 3x^2, -2y\rangle$. $\nabla f|_{(1,2)} = \langle 3, -4\rangle$.

$f$ increases most rapidly in the direction $\frac{\nabla f}{|\nabla f|} = \frac{\langle 3,-4\rangle}{\sqrt{3^2+4^2}} = \frac{\langle 3,-4\rangle}{5}$

decreases - - - - - - - - $-\frac{\nabla f}{|\nabla f|} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$.

direction of zero change: $\underline{\langle \frac{4}{5}, \frac{3}{5}\rangle, \langle -\frac{4}{5}, -\frac{3}{5}\rangle}$

Geometric meaning of $\nabla f$. $f = f(x,y)$




level curve $\boxed{f(x,y)=C}$. $\underline{x = x(t),\ y = y(t)}$.

$f(x(t), y(t)) \equiv C$, $\forall t$. $\vec{r}(t) = \langle x(t), y(t)\rangle$

$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$

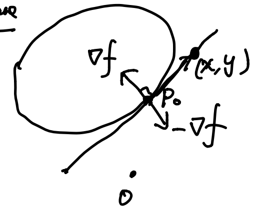
$\langle f_x, f_y\rangle \bullet \langle x', y'\rangle = 0 \iff \boxed{\nabla f \perp \vec{r}\,'(t)}$

$\langle f_x, f_y\rangle = \nabla f$, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$

tangent vector the level curve

$\nabla f \perp \vec{r}\,'(t)$: This means that $\nabla f$ is perpendicular to level curves.

Over the mountain, streams flow perpendicular to the contour $\underline{\text{in the direction of steepest descent}}$. $-\frac{\nabla f}{|\nabla f|}$.




$P_0 = (x_0, y_0)$. $\nabla f = \langle f_x, f_y\rangle$.

$\langle x - x_0, y - y_0\rangle \bullet \langle f_x, f_y\rangle|_{P_0} = 0$.

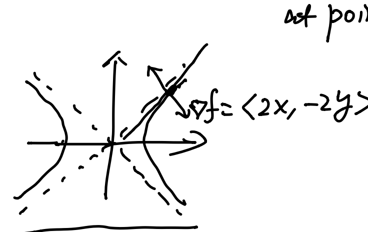
$$\boxed{f_x(P_0)\cdot(x-x_0) + f_y(P_0)\cdot(y-y_0) = 0}$$

tangent line to the level curve $f(x,y)=C$ at point $P_0(x_0, y_0)$.

Ex: $\underline{f(x,y) = x^2 - y^2}$.

level curve $\underline{f(x,y) = 1} = x^2 - y^2$.

$y^2 + 1 = x^2 \geq 1$

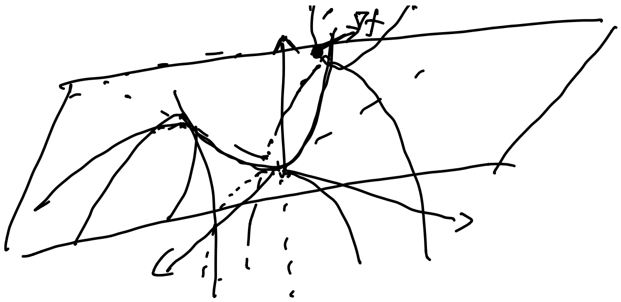



$P_0(2, \sqrt{3})$.

$\nabla f|_{P_0} = \langle 4, -2\sqrt{3}\rangle$

tangent line: $4\cdot(x-2) + (-2\sqrt{3})\cdot(y - \sqrt{3}) = 0$.

$4x - 2\sqrt{3}y = 8 - 6 = 2 \iff \boxed{2x - \sqrt{3}y = 1}$




$\nabla f = \langle 2x, -2y \rangle.$

$1 = x^2 - y^2, \quad y = \sqrt{x^2-1}$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}} = k$

$\boxed{y - y_0 = k \cdot (x - x_0).}$

Algebraic rules for gradients.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$,

$\nabla(f \cdot g) = \underline{(\nabla f) \cdot g + f \cdot (\nabla g)}.$

$\nabla(f \cdot g) = \langle \frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg) \rangle$

$\nabla\left(\frac{f}{g}\right) = \nabla\left(f \cdot \frac{1}{g}\right) = \nabla f \cdot \frac{1}{g} + \underbrace{f \nabla\left(\frac{1}{g}\right)}_{= f \cdot \left(-\frac{1}{g^2}\right) \nabla g} = \nabla f \cdot \frac{1}{g} - \frac{f}{g^2} \nabla g = \frac{(\nabla f) \cdot g - f(\nabla g)}{g^2}$

- Functions of 3-variables $w = f(x, y, z)$. $\nabla f = \langle f_x, f_y, f_z \rangle$

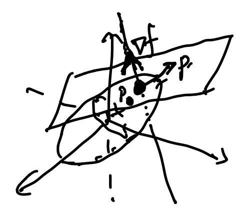
$D_{\vec{u}} f = \nabla f \bullet \vec{u}$ ← dot product

$\vec{u} \bullet \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$

$f(x, y, z) = C$ level surface in $\mathbb{R}^3$ ← 3-dim space.

$\nabla f|_{P_0}$ is perpendicular to the level surface at any point on the level surface.

Ex: $f(x, y, z) = x^2 + 2y^2 + 3z^2$.




$S: \quad f(x, y, z) = 1 = x^2 + 2y^2 + 3z^2.$

$1 = x^2 + \frac{y^2}{\frac{1}{2}} + \frac{z^2}{\frac{1}{3}}$

$\nabla f = \langle 2x, 4y, 6z \rangle = \langle f_x, f_y, f_z \rangle.$ $\quad P_0 = (x_0, y_0, z_0) \in S$

tangent plane at $P_0$: $\nabla f|_{P_0} \bullet \underbrace{\langle x - x_0, y - y_0, z - z_0 \rangle}_{= \overrightarrow{P_0P}} = 0.$

$$\boxed{f_x(P_0)(x - x_0) + f_y(P_0) \cdot (y - y_0) + f_z(P_0) \cdot (z - z_0) = 0.}$$

<u>normal line</u>: $\vec{r}(t) = \overrightarrow{OP_0} + t \cdot \nabla f(P_0)$

$x(t) = x_0 + f_x(P_0) \cdot t$, $y(t) = y_0 + f_y(P_0) \cdot t$, $z(t) = z_0 + f_z(P_0) \cdot t$.