$z = f(x,y)$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}.$$

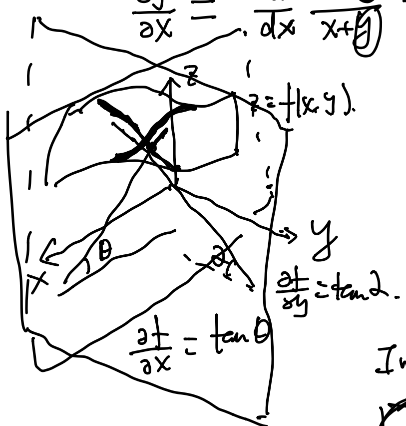
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0}$$

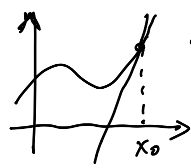
$$\frac{d}{dx}\frac{f}{g} = \frac{\left(\frac{d}{dx}f\right)g - f\cdot\frac{d}{dx}g}{g^2}$$

Ex: $f(x,y) = \dfrac{2x-y}{x+y}$.

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\frac{2x-y}{x+y} = \frac{\frac{\partial}{\partial x}(2x-y)\cdot(x+y) - (2x-y)\frac{\partial}{\partial x}(x+y)}{(x+y)^2} = \frac{2\cdot(x+y) - (2x-y)\cdot 1}{(x+y)^2}$$

$$= \frac{3y}{(x+y)^2}$$





$y = f(x)$

$f'(x_0)$ = slope of the tangent line

= rate of change of f at x_0.

Implicit differentiation:

$$\boxed{F(x,y,z)=0.} \quad \text{implicit}$$

$$\Downarrow$$

$$z = z(x,y)$$

$$\boxed{x\cdot y + \cos(x+z) = 1.}$$

$$\Downarrow$$

$$\underline{z = z(x,y).}$$

$$\underline{F(x,y,z(x,y)) = 0} \Rightarrow \underline{\frac{\partial}{\partial x} F(x,y,z(x,y))} = 0$$

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0.$$

$$\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

$$w = F(x,y,z), \quad \frac{\partial F}{\partial z} = \lim_{h\to 0}\frac{F(x,y,z+h) - F(x,y,z)}{h} = F_z \qquad \underline{F_x = \frac{\partial F}{\partial x},\ F_y = \frac{\partial F}{\partial y}.}$$

Ex: $\underbrace{x\cdot y + \cos(x+z) - 1}_{=F} = 0 \Rightarrow z = z(x,y) = \underline{\cos^{-1}(1-xy) - x}$

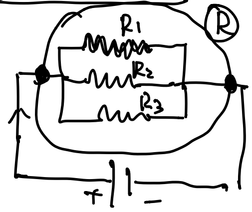
$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y - \sin(x+z)}{-\sin(x+z)} = \frac{y - \sin(x+z)}{\sin(x+z)}}\ \Bigg|_{z=z(x,y)}.$$

$F_x = y - \sin(x+z)$, $F_z = -\sin(x+z)$

$F(x,y)=C \rightsquigarrow y=y(x)$. $\boxed{\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}}$

$F = x\cdot y+\cos(x+z)-1$, $F_x=\dfrac{\partial F}{\partial x}=y-\sin(x+z)$

Ex:



$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} \rightsquigarrow R=R(R_1,R_2,R_3).$$

$$\Rightarrow R=\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

$\dfrac{\partial R}{\partial R_1}$:

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_1}=-\frac{1}{R_1^2}+0+0=-\frac{1}{R_1^2}\Rightarrow \boxed{\frac{\partial R}{\partial R_1}=\frac{R^2}{R_1^2}}$$

$$\frac{d}{dR}\left(\frac{1}{R}\right)\frac{\partial R}{\partial R_1} \qquad \frac{d}{dR}\left(\frac{1}{R}\right)=\frac{d}{dR}R^{-1}=-1\cdot R^{-2}=-\frac{1}{R^2}.$$

- $y=f(x)$, $f'(x_0)$ exists $\Rightarrow$ $f(x)$ is continuous at x_0.

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \Rightarrow \lim_{x\to x_0} f(x)=f(x_0).$$

$z=f(x,y)$. $f_x=\dfrac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$, $f_y=\dfrac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$ exist $\overset{?}{\not\Rightarrow}$ f is continuous at (x_0,y_0).

$$f(x,y)=\begin{cases}0 & xy=0\\ \dfrac{y}{x} & xy\neq 0\end{cases}$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ DNE $\to$ $f(x,y)$ not cont. at $(0,0)$

$$\frac{\partial f}{\partial x}\Big|_{(0,0)}=\frac{d}{dx}f(x,0)\Big|_{x=0}=\frac{d}{dx}0\Big|_{x=0}=0.$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)}=\frac{d}{dy}f(0,y)\Big|_{y=0}=\frac{d}{dy}0\Big|_{y=0}=0$$

partial.

- 2nd order derivatives

$z=f(x,y)$. $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x}f=f_{xx}$, $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y}f=f_{yy}$.

mixed derivatives: $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=f_{yx}=(f_y)_x$, $\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}=f_{xy}=(f_x)_y$.

Ex: $f(x,y) = y \cdot \ln(x+2y)$.

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(y \cdot \frac{1}{x+2y}\right) = y \cdot -\frac{1}{(x+2y)^2} = -\frac{y}{(x+2y)^2}$.

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{y}{x+2y}\right) = \frac{1 \cdot (x+2y) - y \cdot \frac{\partial}{\partial y}(x+2y)}{(x+2y)^2} = \frac{x+2y-y \cdot 2}{(x+2y)^2} = \frac{x}{(x+2y)^2}$

const.

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial x}\left(1 \cdot \ln(x+2y) + y \cdot \frac{1}{x+2y} \cdot 2\right)$.

$= \frac{1}{x+2y} + 2 \cdot y \frac{\partial}{\partial x}\left(\frac{1}{x+2y}\right) = \frac{1}{x+2y} + 2y \cdot \left(-\frac{1}{(x+2y)^2}\right)$

$= \frac{1}{(x+2y)^2} \cdot (x+2y-2y) = \frac{x}{(x+2y)^2}$.

$\Rightarrow f_{xy} = f_{yx}$.

$\cdot(a,b)$

Thm (Clairaut's thm). If $f, f_x, f_y, f_{xy}, f_{yx}$ are defined near (a,b) and
Mixed derivative Thm
are continuous at (a,b). Then $\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$

Ex: $f(x,y) = x^3 e^{-x} + \sin(xy)$

$f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left(3x^2 e^{-x} - x^3 e^{-x} + \cos(xy) \cdot y\right)$

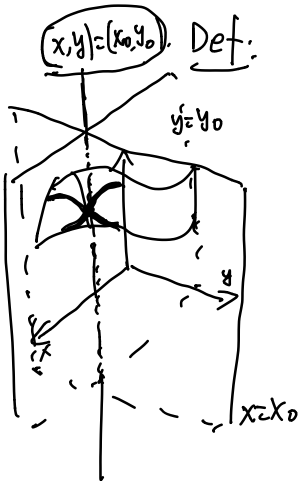
$\parallel$

$f_{yx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}\left(\cos(xy) \cdot x\right) = \cdots$

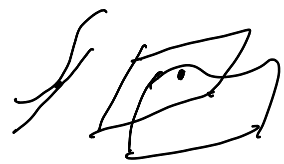
- higher order partial derivatives: $f_{yxx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$

$f_{xyyx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ $\qquad f_{xyx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$

Mixed derivative thm: If f is suff. smooth, then we can switch the order of partial derivatives.



Def: $z=f(x,y)$ is **differentiable** at (x_0,y_0) if $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ ~~exist~~.



$$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)\cdot\underbrace{(x-x_0)}_{\Delta x}+f_y(x_0,y_0)\cdot\underbrace{(y-y_0)}_{\Delta y}$$
$$+\ \varepsilon_1\cdot\Delta x+\varepsilon_2\Delta y$$

s.t. $\varepsilon_1,\varepsilon_2\to 0$ as both $\Delta x,\Delta y\to 0$.

Fact: differentiability $\Rightarrow$ continuity

existence of partial derivatives $\not\Rightarrow$

Thm: Suppose f_x, f_y are defined throughout an open region containing (x_0,y_0), and f_x, f_y continuous at (x_0,y_0). Then f is differentiable at (x_0,y_0)

$\boxed{\text{existence and continuity of partial derivatives}}\Rightarrow$ differentiability $\Rightarrow$ continuity

14.4 Chain rule

$w=f(x)$, $x=x(t)$. $w=f(x(t))=w(t)$. $\dfrac{dw}{dt}=\dfrac{dw}{dx}\cdot\dfrac{dx}{dt}$.

$t\to x\to w$

$w=f(x,y)$. $x=x(t)$, $y=y(t)$. $\leadsto$ $w=f(x(t),y(t))=w(t)$.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}.$$

$$\begin{array}{ccc} & x & \\ t & & w \\ & y & \end{array}$$

$t\nearrow x \searrow w$, $t \searrow y \nearrow w$ dependency diagram

$w=f(x,y,z)$, $x=x(t)$, $y=y(t)$, $z=z(t)$.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$t\to x\to w$, $t\to y\to w$, $t\to z\to w$

$w=f(x,y)$ $x=x(r,s)$, $y=y(r,s)$.

$r, s \to x \to w$, $r, s\to y\to w$.

$w=f(x(r,s),y(r,s))=w(r,s)$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.\qquad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Implicit Differentiation: $F(x,y,z)=0$. $\overset{\text{implicitly}}{\leadsto}$ $z=z(x,y)$ satisfies

$$w = \boxed{F(x,y,z(x,y))} \equiv 0$$

$$0 = \frac{\partial w}{\partial x} = \underbrace{\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial x}}_{=\frac{\partial F}{\partial x}\cdot 1} + \underbrace{\frac{\partial w}{\partial y}\frac{\partial y}{\partial x}}_{=\frac{\partial F}{\partial y}\cdot 0} + \underbrace{\frac{\partial w}{\partial z}\frac{\partial z}{\partial x}}_{=\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}}$$

$$\begin{tikzcd} x \arrow[r] & x \arrow[dr] & \\ y \arrow[r] \arrow[dr] & y \arrow[r] & w. \\ & z \arrow[ur] & \end{tikzcd}$$

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = F_x + F_z\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$