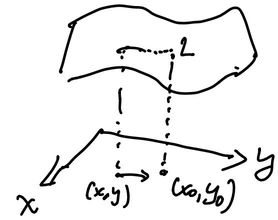
**limit**: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $\forall \varepsilon>0, \exists \delta>0$, s.t. if $0<|(x-x_0, y-y_0)|<\delta$ then $|f(x,y)-f(x_0,y_0)|<\varepsilon$

$|(x-x_0, y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$

$\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0, \quad \lim\limits_{(x,y)\to(x_0,y_0)} y = y_0$




**Properties**: $\lim f\pm g = \lim f \pm \lim g$

......

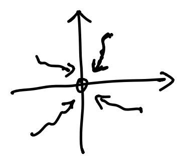
**Ex**: $\lim\limits_{(x,y)\to(1,2)} \frac{2x-y}{x-3y} = \frac{\lim(2x-y)}{\lim(x-3y)} = \frac{2\cdot\lim x - \lim y}{\lim x - 3\cdot \lim y} = \frac{2\cdot 1-2}{1-3\cdot 2} = 0.$ $\quad (1-3\cdot 2 \neq 0.)$

undefined at (1,2)

**Ex**: $\lim\limits_{(x,y)\to(1,2)} \frac{2x-y}{4x^2-y^2} = \lim\limits_{(x,y)\to(1,2)} \frac{\cancel{2x-y}}{(2x+y)\cancel{(2x-y)}} = \lim\limits_{(x,y)\to(1,2)} \frac{1}{2x+y} = \frac{1}{2\cdot 1+2} = \frac{1}{4}.$

$\left(\lim\limits_{(x,y)\to(1,2)} (4x^2-y^2) = 4\cdot 1^2-2^2 = 0\right)$

**Ex**: $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x}$

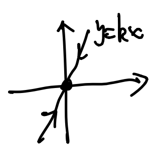


If limit exists, then the limit does NOT depend on how $(x,y)$ approaches $(x_0,y_0)$

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \rightsquigarrow \lim\limits_{\substack{(x(t),y(t))\to(x_0,y_0)\\ (x_n,y_n)\to(x_0,y_0)}} f(x_n,y_n)$ should exist and should be the same for all ways of approaching $(x_0,y_0)$

$k\in\mathbb{R}$

$\boxed{y=k\cdot x}$ $\quad \lim\limits_{(x,y)\to(0,0)} \frac{kx}{x} = \lim\limits_{(x,y)\to(0,0)} k = k$ $\quad$ the original limit DNE

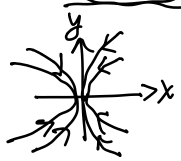



**Ex**: $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+y^2}$ $\quad \overset{y=kx}{} \quad \frac{2x^2-k^2x^2}{x^2+k^2x^2} = \frac{2-k^2}{1+k^2}$ depends on $k \Rightarrow$ the limit DNE

Ex: $\lim_{(x,y)\to(0,0)} \underbrace{\frac{2x-y^2}{x+y^2}}_{f(x,y)}$

$\boxed{y=kx,\quad \frac{2x-k^2x^2}{x+k^2x^2} = \frac{2-k^2x}{1+k^2x} \xrightarrow{(x,y)\to(0,0)} \frac{2-k^2\cdot 0}{1+k^2\cdot 0} = 2.}$

$x = k\cdot y^2$



$f(ky^2,y) = \left(\frac{2\cdot k\cdot y^2 - y^2}{ky^2+y^2}\right) = \frac{2k-1}{k+1}$ depends on $k$ $\Rightarrow$ The limit DNE (does not exist) $\lim_{(x,y)\to(0,0)} \frac{2x-y^2}{x+y^2}$

$\boxed{x=ky}$ $\quad \frac{2\cdot ky - y^2}{ky+y^2} = \frac{2k-y}{k+y} \xrightarrow{(x,y)\to(0,0)} \frac{2k}{k} = 2$ $\not\Rightarrow$ limit exists.

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

$\boxed{\lim_{(x,y)\to(x_0,y_0)} f = L \iff \lim_{(x,y)\to(x_0,y_0)} |f-L| = 0}$

$0 \le \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2\cdot|y|}{x^2+y^2} = \frac{x^2}{x^2+y^2}|y| \le 1\cdot|y| = |y|$

$\left|\frac{x^2y}{x^2+y^2} - 0\right|$, $\quad \frac{x^2}{x^2+y^2} \le 1$, $\quad 0 \to 0$, $\quad |y| \to 0$

Squeezing thm: $g(x,y) \le f(x,y) \le h(x,y)$. $\quad g \to L,\ h \to L \Rightarrow f \to L$

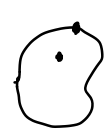
$\Rightarrow \lim \left|\frac{x^2y}{x^2+y^2}\right| = 0 \Rightarrow \boxed{\lim \frac{x^2y}{x^2+y^2} = 0.}$

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^3+y^3}$ $\quad$ let $y=kx$. $\frac{x^2}{x^3+k^3x^3} = \frac{1}{(1+k^3)\cdot x} \xrightarrow{x\to 0}$ DNE

$\Rightarrow \lim \frac{x^2}{x^3+y^3}$ DNE

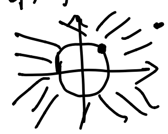
- continuity of function

Def: $f(x,y)$ is continuous at $(x_0,y_0)$ if
- $(x_0,y_0) \in Dom(f)$ (i.e. $f$ is defined at $(x_0,y_0)$).
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
- $\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)}$



$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

Ex: $\underline{f(x,y)=\sqrt{x^2+y^2-1}}$ $\quad Dom(f)=\{x^2+y^2\geq 1\}$



$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{x^2+y^2-1} = \sqrt{\lim_{(x,y)\to(x_0,y_0)} (x^2+y^2-1)}$$

$(x_0,y_0) \in Dom(f)$

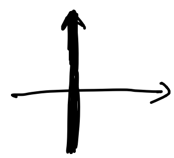
$$= \sqrt{x_0^2+y_0^2-1} = \underline{f(x_0,y_0)}$$

is continuous at any pt. in $Dom(f)$.

In General, polynomial functions, rational functions are continuous at any point in their domain.

Ex: $f(x,y)=\begin{cases} \frac{y}{x} & \text{if } x\neq 0 \\ C & \text{if } x=0. \end{cases}$

$C$: constant



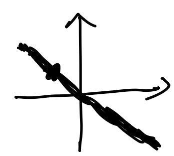
- $f$ is continuous on $\mathbb{R}^2\setminus\{x=0\}$.
- $f$ is not continuous at any point on the $y$-axis. i.e. any point $(0,y_0)$

$\lim_{(x,y)\to(0,y_0)} \frac{y}{x}$ DNE

case 1: $y_0\neq 0$. $\lim_{(x,y)\to(0,y_0)} \frac{y}{x}$, $y\to y_0\neq 0$, $x\to 0$ DNE

case 2: $y_0=0$. $\lim_{(x,y)\to(0,0)} \frac{y}{x}$ DNE

Ex: $f(x,y)=\begin{cases} \frac{x^2}{x+y} & x+y\neq 0 \\ 1 & x+y=0 \end{cases}$



- $f(x,y)$ is continuous away from the line $x+y=0$. $\lim_{(x,y)\to(x_0,y_0)} \frac{x^2}{x+y} = \frac{x_0^2}{x_0+y_0}$.
- case 1: $x_0\neq 0$, $x_0+y_0=0$. $\lim_{(x,y)\to(x_0,y_0)} \frac{x^2}{x+y}$. $\lim_{(x,y)\to(x_0,y_0)} \left|\frac{x^2}{x+y}\right| = \frac{x_0^2}{0} = +\infty$ $\Rightarrow \lim$ DNE.

case 2: $x_0=0$. $x_0+y_0=0$ $\Rightarrow x_0=y_0=0$. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x+y}$

$y=kx$: $\frac{k^2x^2}{x+kx} = \frac{k^2}{1+k}\cdot x \xrightarrow{x\to 0} 0$

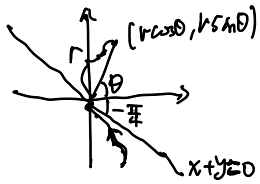
$y=kx^2$: $\frac{x^2}{x+kx^2} = \frac{x}{1+kx} \xrightarrow{x\to 0} 0$.

$x=ky^2$: $\frac{k^2y^4}{ky^2+y} = \frac{k^2y^3}{ky+1} \xrightarrow{y\to 0} 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x+y}$$

$\boxed{(x,y)\to(0,0) \Leftrightarrow r\to 0}$

$x = r\cos\theta$
$y = r\sin\theta$




$\cancel{\left|\frac{x}{x+y}\right| \le 1}$

$$\frac{r^2\cos^2\theta}{r(\cos\theta+\sin\theta)} = r\,\frac{\cos^2\theta}{\cos\theta+\sin\theta}$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha=\theta,\ \beta=\frac{\pi}{4}$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt2}{2} = \frac{1}{\sqrt2}$

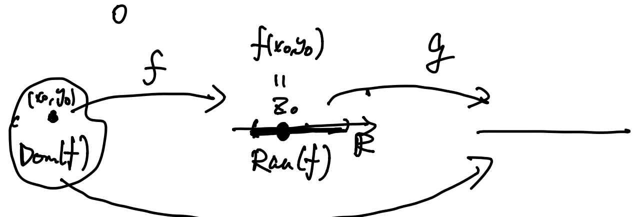
$$r\,\frac{\cos^2\theta}{\sqrt2\,\sin(\theta+\frac{\pi}{4})} \longrightarrow k$$

$\Rightarrow$ limit DNE

$$r_n = \frac{k\sqrt2\,\sin(\theta_n+\frac{\pi}{4})}{\cos^2\theta_n}$$

$\theta_n \to -\frac{\pi}{4}$, $r_n \to 0$

- Continuity of composition




$$g\circ f\,(x,y) = g(f(x,y)).$$

If $f$ is continuous at $(x_0,y_0)$ and $g$ is continuous at $f(x_0,y_0)$
then $g\circ f$ is continuous at $(x_0,y_0)$.

## 14.3 Partial Derivatives

$z = f(x,y)$

Def: partial derivative of $f(x,y)$ with respect to $x$ at point $(x_0,y_0)\in \mathrm{Dom}(f)$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \left.\frac{d}{dx}f(x,y_0)\right|_{x=x_0}.$$

$$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = \frac{\partial z}{\partial x}(x_0,y_0)$$

Ex: $f(x,y) = x\cdot\sin(xy)$

$$f_x = \frac{\partial f}{\partial x} = 1\cdot\sin(xy) + x\cdot\cos(xy)\cdot y = \sin(xy) + xy\cos(xy).$$

$$f_y = \frac{\partial f}{\partial y} = x\cdot\cos(xy)\cdot x = x^2\cdot\cos(xy)$$