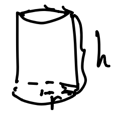
14.1.



$V = \pi r^2 \cdot h = V(r, h)$

$v = v(s, T)$

speed of sound

D : a set of n-tuples of real numbers $\quad \bar{x} = (x_1, \cdots, x_n) \in D$

$\bar{x} \longmapsto w = f(x_1, \cdots, x_n) = f(\bar{x})$

dependent variable   independent variables

$z = f(x, y)$

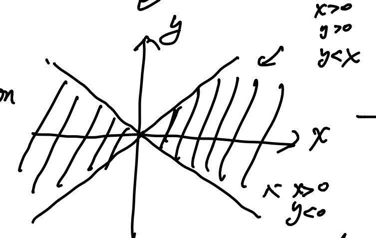
$w = f(x, y, z)$.

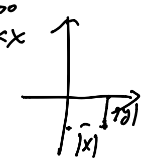
$y^2 \le x^2 \Leftrightarrow \boxed{|y| \le |x|}$

Ex: $z = \sqrt{x^2 - y^2}$ is defined only when $x^2 - y^2 \geq 0$

Domain: $\boxed{D = \{(x,y) ;\ x^2 - y^2 \geq 0\}}$

Domain of a fct: set of n-tuples for which the function is defined





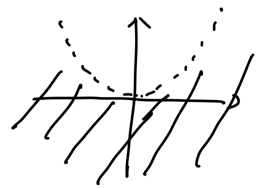
Range of a fct: set of values taken by a function

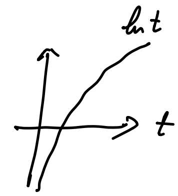
Range: $[0, +\infty)$.

Ex: $z = \ln(x^2 - y) = \log_e (x^2 - y)$

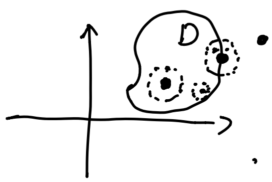
Domain: $x^2 - y > 0 \Leftrightarrow y < x^2$

Range: $(-\infty, +\infty)$.



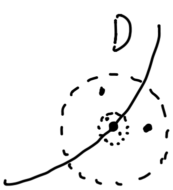


Def:



• Def. P is an $\boxed{\text{interior point}}$ of D if $\exists$ $r > 0$ s.t. a disk of radius r centered at P is contained in D

• P is called a $\boxed{\text{boundary point}}$ if: any disk (of positive radius) centered at P contains both pt. in D and a point not in D

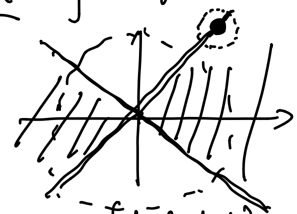


$\exists$: there exists	$\forall$: For any / For all	s.t. such that

Def: · A set D is <u>open</u> if it only consists of interior points.

[Set topology] · A set D is <u>closed</u> if it contains all its boundary points

Ex: $D = \{ |y| \leq |x| \}$



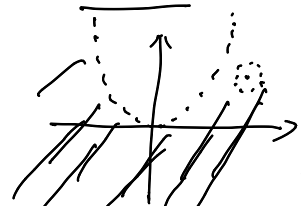
unbounded

interior of D '' set of interior points $\rightarrow$ $\mathring{D} = \{ y < x \}$

$\partial D = \{ |y| = |x| \} = \{ y = x \} \cup \{ y = -x \}$

$\partial D \subset D \Rightarrow D$ is <u>closed</u> not open

Ex: $D = \{ y < x^2 \}$



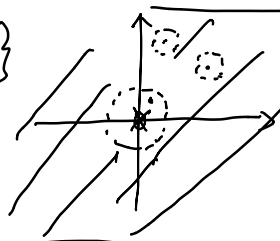
not closed.

$\mathring{D} = D \Rightarrow D$ is open.

$\partial D = \{ y = x^2 \} \not\subset D$.

∂D: set of boundary points

Ex: $D = \mathbb{R}^2 \setminus \{(0,0)\}$

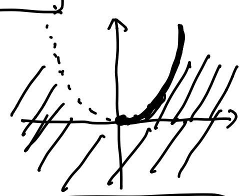


$\mathring{D} = D$ open set

$\boxed{\partial D = \{(0,0)\}}$

high dim		1-dim
open set	$\Longleftrightarrow$	open interval
closed set	$\Longleftrightarrow$	closed interval
neither open nor closed	$\Longleftrightarrow$	$[a, b)$

Ex:

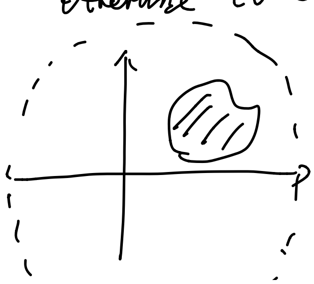


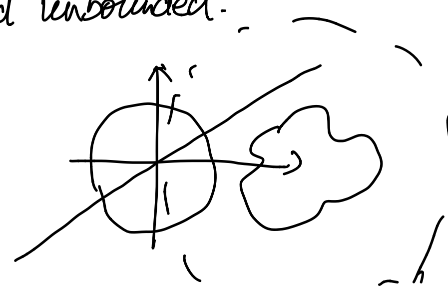
$\{ y < x^2 \} \cup \{ y = x^2, x \geq 0 \} = D$

$\mathring{D} = \{ y < x^2 \}$, $\partial D = \{ y = x^2 \}$

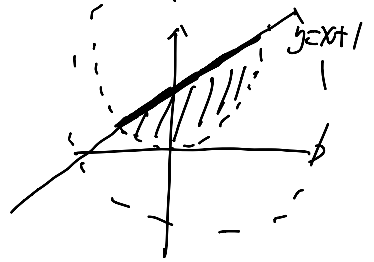
neither open nor closed.

Def: A region is bounded if it is contained in a disk of finite radius otherwise it is called unbounded.





Ex: $D=\{y\geq x^2\}\cup\{y\leq x+1\}$



bounded, neither open nor closed.

$\{P\in\mathbb{R}^2 ; P\notin D\}$ complement of D.

$\quad \| $

Fact: · A region D is closed $\Longleftrightarrow$ D^c is open

$\boxed{\mathring{\phi}=\phi, \partial\phi=\phi}$ · $\emptyset$ the empty set is both open and closed

$\mathring{(\mathbb{R}^2)}=\mathbb{R}^2 \longrightarrow$ $\mathbb{R}^2$ entire plane is both open and closed.

$\partial\mathbb{R}^2=\phi$

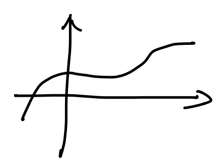
ϕ and $\mathbb{R}^2$ are the only ~~region~~ subsets that are both open and closed.

$$\boxed{\begin{array}{l}\mathring{\phi}=\phi \Rightarrow \phi \text{ is open}\\ \boxed{\partial\phi=\phi} \Rightarrow \phi \text{ is closed}\end{array}}$$

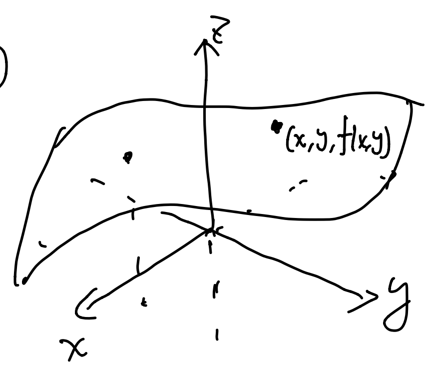
$P \quad\quad \phi$

P

$y=f(x)$



$z=f(x,y)$

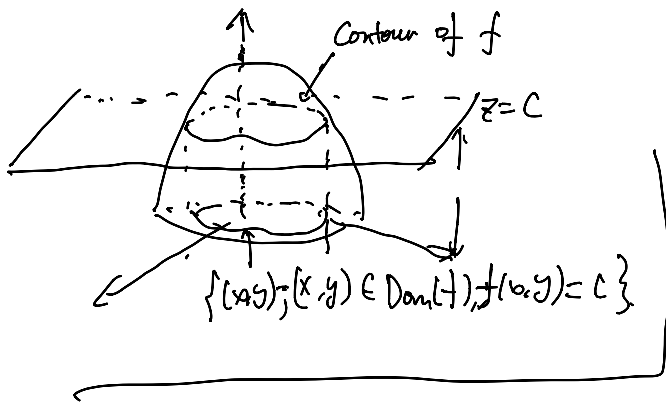


graph $=\{(x,y,f(x,y)) ; (x,y)\in D\}$

D ↑ domain of f.

$\mathrm{Dom}(f)=$ domain of $f=$ set of pts where f is defined

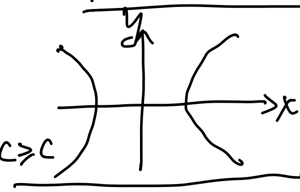
level curve of $z=f(x,y)$ at level c: $\{(x,y)\in\mathbb{R}^2 ; (x,y)\in \mathrm{Dom}(f), f(x,y)=c\}$



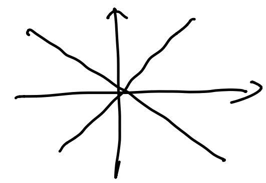
Ex: $f(x,y) = x^2 - y^2$ $\quad$ $Dom(f) = \mathbb{R}^2$

Level curves $\{(x,y);\ x^2-y^2 = c\}$

$c>0$: $x^2 = y^2 + c \geq c$

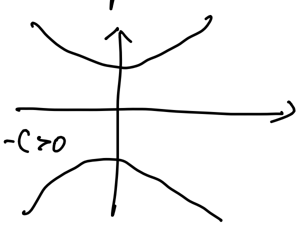


$c=0$: $x^2-y^2=0 \Leftrightarrow$ $x-y=0$, $x+y=0$

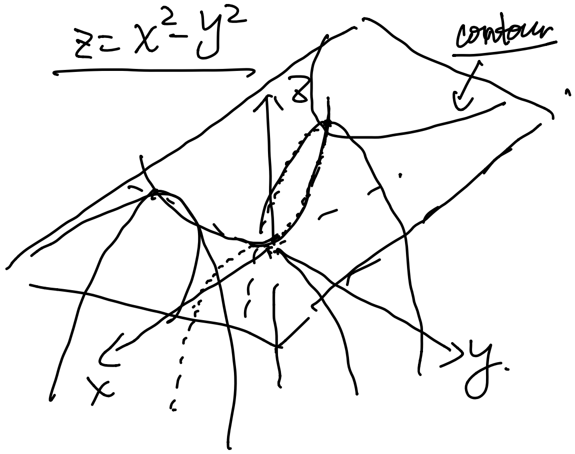


$c<0$

$y^2 = x^2 - c \geq -c > 0$



$z = x^2 - y^2$



Ex: $w = f(x,y,z)$

$w = \sqrt{x^2+2y^2+3z^2}$ $\qquad$ $Dom(f) = \mathbb{R}^3 = \{(x,y,z) \in \mathbb{R}^3\}$

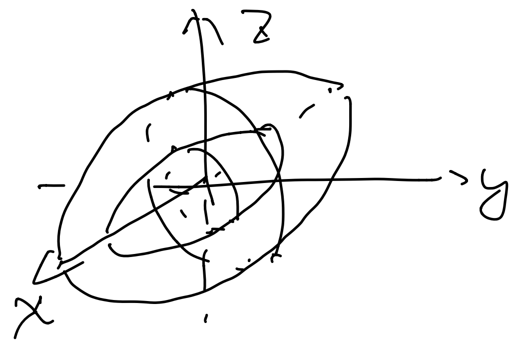
range of f = $Ran(f) = [0, +\infty)$.

(interior pts $\quad$ use balls (instead of disks).
boundary pts)

graph of $w = f(x,y,z)$

Level surface : $\{(x,y,z) \in Dom(f);\ f(x,y,z) = c\}$
at level c

$$\sqrt{x^2+2y^2+3z^2} = c \Longleftrightarrow x^2+2y^2+3z^2 = c^2 \Leftrightarrow \boxed{\frac{x^2}{c^2} + \frac{y^2}{\left(\frac{c^2}{2}\right)} + \frac{z^2}{\left(\frac{c^2}{3}\right)} = 1}$$



14.2. Limits

Def: $z = f(x,y)$. We say that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

epsilon, delta

if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $|f(x,y) - L| < \varepsilon$ whenever $0 < |(x,y) - (x_0,y_0)| < \delta$.

(ε-δ language): for any $\varepsilon > 0$, there exists, such that

$$|(x,y)-(x_0,y_0)| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Ex: $\boxed{\lim_{(x,y)\to(x_0,y_0)} \underset{f(x,y)}{\underset{\shortparallel}{x}} = x_0}$, $\lim_{(x,y)\to(x_0,y_0)} y = y_0$

Fix any $\varepsilon > 0$,

$$\underset{|f(x,y)-f(x_0,y_0)|}{\underset{\shortparallel}{\boxed{|x - x_0|}}} < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon$$

we can choose $\delta = \varepsilon$.

Rule of limits:

$$\boxed{\begin{aligned} &\lim (f(x,y) + g(x,y)) = (\lim f) + (\lim g) \\ &\lim (f \cdot g) = (\lim f) \cdot (\lim g) \\ &\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad \text{as long as } (\lim g \neq 0) \end{aligned}}$$