13.2 $\quad \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

indefinite integral $\quad \int \vec{r}(t) = \int f(t)dt\,\vec{i} + \int g(t)dt\,\vec{j} + \int h(t)dt\,\vec{k}$

$= \vec{R}(t) + \vec{C}$ $\quad$ vector valued function.

primitive function ($\vec{R}$), constant vector ($\vec{C}$)

$\vec{R}'(t) = \vec{r}(t)$.

definite integral $\quad \int_a^b \vec{r}(t)dt = \int_a^b f(t)dt\,\vec{i} + \cdots\vec{j} + \cdots\vec{k}$ $\quad$ vector.

position function $\vec{r}(t) \rightsquigarrow$ velocity $\vec{v}(t) = \vec{r}'(t) \rightsquigarrow$ acceleration $\vec{a}(t) = \vec{v}'(t)$

$\vec{r}(t) = \int_{t_0}^{t} \vec{v}(\tau)d\tau + \vec{r}(t_0) \leftarrow \boxed{\vec{v}(t) = \int_{t_0}^{t} \vec{a}(\tau)d\tau + \vec{v}(t_0)} \leftarrow \underline{a(t)}$

$\hookrightarrow \begin{cases} \vec{r}(t_0) = \vec{r}(t_0) \\ \vec{r}'(t) = \vec{v}(t). \end{cases}$ $\quad \hookrightarrow \begin{cases} \vec{v}(t_0) = \vec{v}_0 \\ \vec{v}'(t) = \vec{a}(t) \end{cases}$ $\quad \frac{d}{dt}\int_{t_0}^{t} f(\tau)d\tau = f(t).$

Ex: $\quad \vec{r}(0) = \langle -1, 2\rangle, \quad \vec{v}(t) = 3t^2\vec{i} + \cos(\pi t)\vec{j}$

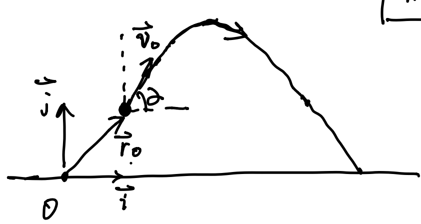
$\Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau)d\tau = \langle -1, 2\rangle + \int_0^t \left(3t^2\vec{i} + \cos(\pi t)\vec{j}\right)dt$

$= \langle -1, 2\rangle + \left(t^3\vec{i} + \frac{1}{\pi}\sin(\pi t)\vec{j}\right)\Big|_0^t$

$= \langle -1, 2\rangle + \langle t^3, \frac{1}{\pi}\sin(\pi t)\rangle = \langle -1 + t^3, 2 + \frac{1}{\pi}\sin(\pi t)\rangle$

Application: projectile motion $\quad g = 9.8\,m/s^2?$

$\boxed{m\cdot\vec{a} = \vec{G} = -m\cdot g\cdot\vec{j}} \Rightarrow \underline{\vec{a} = -g\cdot\vec{j}}$



$\rightsquigarrow \vec{v}(t) = \vec{v}_0 + \int_0^t \underbrace{\vec{a}(\tau)}_{=-g\cdot\vec{j}}d\tau = \underline{\vec{v}_0 - g t\cdot\vec{j}}$

$\rightsquigarrow \vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(\tau)d\tau = \vec{r}_0 + \int_0^t \left(\vec{v}_0 - g\tau\vec{j}\right)d\tau$

$= \vec{r}_0 + \vec{v}_0\cdot t - \frac{1}{2}g t^2\cdot\vec{j}$.

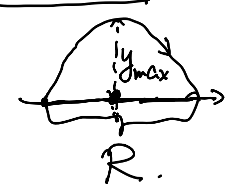
$v_0 = |\vec{v}_0|$ speed

$$\langle x(t), y(t)\rangle = \langle x_0, y_0\rangle + \langle v_0\cdot\cos\alpha, v_0\cdot\sin\alpha\rangle\cdot t - \frac{1}{2}g t^2\cdot\langle 0,1\rangle$$

$$= \langle \underbrace{x_0 + v_0\cdot\cos\alpha\cdot t}_{=x}, \underbrace{y_0 + v_0\cdot\sin\alpha t - \frac{1}{2}g t^2}_{=y}\rangle. \qquad t \geq 0.$$

$$t = \frac{x-x_0}{v_0\cdot\cos\alpha} \rightsquigarrow y - y_0 = v_0\cdot\sin\alpha\cdot\frac{x-x_0}{v_0\cos\alpha} - \frac{1}{2}g\cdot\left(\frac{x-x_0}{v_0\cos\alpha}\right)^2$$

For simplicity, assume $(x_0, y_0) = (0,0)$: $\boxed{y = (\tan\alpha)\cdot x - \frac{g}{2v_0^2\cos^2\alpha}x^2}$. (a parabola)



- time of maximal height:

$$\vec{v}(t) = \vec{v_0} - gt\cdot\vec{j} = \langle v_0\cos\alpha, v_0\sin\alpha - gt\rangle$$

$$\boxed{t_1 = \frac{v_0\cdot\sin\alpha}{g}}$$

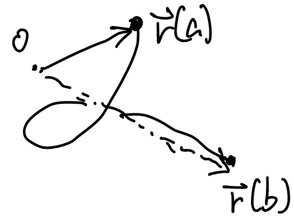
- $$y_{max} = v_0\cdot\sin\alpha\cdot\frac{v_0\cdot\sin\alpha}{g} - \frac{1}{2}\cdot g\cdot\frac{(v_0\sin\alpha)^2}{g^2}$$

$$= \frac{v_0^2\sin^2\alpha}{g} - \frac{1}{2}\frac{v_0^2\sin^2\alpha}{g} = \boxed{\frac{v_0^2\sin^2\alpha}{2g}}.$$

- Range $= (v_0\cdot\cos\alpha)\cdot t_1\cdot 2 = v_0\cdot\cos\alpha\cdot\frac{v_0\sin\alpha}{g}\cdot 2$

$$\boxed{R = \frac{v_0^2\cdot\sin(2\alpha)}{g}}$$

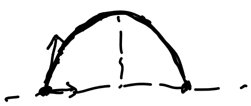
13.3 Arc Length



$\langle x(t), y(t), z(t)\rangle = \vec{r}(t)$, $a \le t \le b$

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2}\,dt$$

EX: $\vec{r}(t) = \langle v_0 \cos\alpha \cdot t,\ v_0 \sin\alpha \cdot t - \frac{1}{2} g t^2 \rangle$



$\alpha = 60^\circ = \frac{\pi}{3}$ $\quad \cos\alpha = \frac{1}{2}$, $\sin\alpha = \frac{\sqrt{3}}{2}$, $\quad v_0 = 1$, $g = 2$

$$= \langle \tfrac{1}{2} t,\ \tfrac{\sqrt{3}}{2} t - t^2 \rangle$$

$$\vec{r}\,'(t) = \langle \tfrac{1}{2},\ \tfrac{\sqrt{3}}{2} - 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2} - 2t\right)^2} = \frac{1}{2}\sqrt{1 + (\sqrt{3} - 4t)^2}$$

$$L = \int_0^{t_{max}} |\vec{r}\,'(t)|\,dt = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{2}\sqrt{1 + (\sqrt{3} - 4t)^2}\,dt.$$

$$\boxed{\begin{array}{l} t_{max} = \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} t_{max} - t_{max}^2 = 0 \end{array}}$$

$\left(x = \sqrt{3} - 4t,\ t = \frac{\sqrt{3} - x}{4}\quad dt = -\frac{dx}{4} \right)$.

$$= \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{3} - 4\times\frac{\sqrt{3}}{2}} \sqrt{1 + x^2}\left(-\frac{dx}{4}\right)$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} = \int_{-\sqrt{3}}^{0} + \int_0^{\sqrt{3}}$$

$$= -\frac{1}{2}\int_{\sqrt{3}}^{-\sqrt{3}} \sqrt{1 + x^2}\,\frac{dx}{4} = \frac{1}{4}\int_0^{\sqrt{3}} \sqrt{1 + x^2}\,dx$$

$x = \tan\theta$, $\quad 1+x^2 = 1+\tan^2\theta = \sec^2\theta$, $\quad dx = d\frac{\sin\theta}{\cos\theta}$

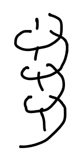
$= 1 + \frac{\sin^2}{\cos^2\theta} = \sec^2\theta$

$$\frac{1}{4}\cdot\int_0^{\frac{\pi}{3}} \sec\theta\cdot\sec^2\theta\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} \underline{\sec^3\theta}\cdot d\theta$$

$$\boxed{\int\sqrt{1+x^2}\,dx = \left(\frac{1}{2}\sqrt{x^2+1}\cdot x + \sinh^{-1}(x).\right)}$$

$$\frac{d\theta}{\cos^3\theta} = \frac{\cos\theta\, d\theta}{\cos^4\theta} = \boxed{\frac{d\sin\theta}{(1-\sin^2\theta)^2}} = \frac{dy}{(1-y^2)^2}$$

Ex: helix : $\boxed{\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t\rangle}$

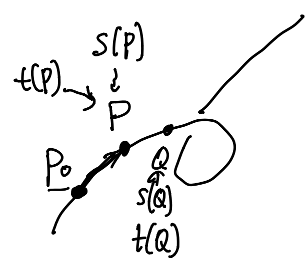


$\vec{r}\,'(t) = \underline{\langle -3\cdot\sin(3t), 3\cos(3t), 4\rangle}$

$$|\vec{r}\,'(t)| = \sqrt{(-3)^2\sin^2(3t) + 3^2\cos^2(3t) + 4^2} = \sqrt{25} = 5.$$

$$\int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b 5\,dt = 5\cdot(b-a).$$

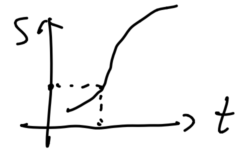
· <u>Arclength parametrization</u> $\quad \vec{r} = \vec{r}(t).$



$$\underline{s = \int_{t_0}^{t} |\vec{r}\,'(\tau)|\,d\tau = s(t)} \rightarrow \frac{ds}{dt} = |\vec{r}\,'(t)| > 0$$

s is an <u>increasing function of t</u> $\quad$ assume $\vec{r}(t)$ smooth $\rightarrow \vec{r}\,'(t) \neq 0$

$\rightsquigarrow$ solve t in terms of s $\quad t = t(s)$



$\rightsquigarrow$ $\vec{r}(t(s)) = \vec{r}(s)$ arc length parametrization.

$\underline{\text{Ex}}$: $|\vec{r}\,'(t)| = 5$ let $t_0 = 0$

$$\underline{s} = \int_0^t 5\,d\tau = \underline{5t} \rightarrow t = \frac{s}{5}$$

$$\vec{r}(t(s)) = \left\langle \cos\left(3 \cdot \frac{s}{5}\right), \sin\left(3 \cdot \frac{s}{5}\right), 4 \cdot \frac{s}{5} \right\rangle$$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{3}{5}s\right), \sin\left(\frac{3}{5}s\right), \frac{4}{5}s \right\rangle}$$

arclength parametrization. $\underline{\left(\begin{array}{l}\text{canonical} \\ \text{intrinsic param.}\end{array}\right)}$

$$\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$$

$$\boxed{\vec{r}(u) = \langle \cos(3u^3), \sin(3u^3), 4u^3 \rangle}$$

$t = u^3$ $\quad$ $\vec{r}(s) = \left\langle \cos\left(\frac{3}{5}s\right), \sin\left(\frac{3}{5}s\right), \frac{4}{5}s \right\rangle$.

$$\underline{\vec{r} = \vec{r}(t)} \rightsquigarrow \underline{\vec{r}(s) = \vec{r}(t(s))} \qquad s \rightarrow t \rightarrow \vec{r}$$

$$\frac{d}{ds}\vec{r}(s) = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{1}{|\vec{r}\,'(t)|} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad \text{unit vector!}$$

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(\tau)|\,d\tau \qquad \frac{ds}{dt} = |\vec{r}\,'(t)| \rightsquigarrow \frac{dt}{ds} = \frac{1}{|\vec{r}\,'(t)|}$$

$$\frac{d\vec{r}}{ds} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$ unit tangent vector at $t = t(s)$

$$= \frac{\vec{r}\,'(t(s))}{|\vec{r}\,'(t(s))|} \rightsquigarrow \boxed{\left|\frac{d\vec{r}}{ds}\right| \equiv 1}$$

Ex: $\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$

$$\vec{r}(s) = \langle \cos(\tfrac{3}{5}t), \sin(\tfrac{3}{5}t), \tfrac{4}{5}s \rangle$$

$$\frac{d\vec{r}}{ds} = \langle -\tfrac{3}{5}\sin(\tfrac{3}{5}t), \tfrac{3}{5}\cos(\tfrac{3}{5}t), \tfrac{4}{5} \rangle$$

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\tfrac{3}{5}\right)^2 \cdot (\sin^2 + \cos^2) + \left(\tfrac{4}{5}\right)^2} = 1.$$