

13.2

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

indefinite
integral

$$\int \vec{r}(t) dt = \int f(t) dt \vec{i} + \int g(t) dt \vec{j} + \int h(t) dt \vec{k}$$

$$= \underbrace{\vec{R}(t)}_{\text{primitive function}} + \underbrace{\vec{C}}_{\text{constant vector}} \quad \text{vector valued function.}$$

$$\vec{R}'(t) = \vec{r}(t).$$

definite
integral

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t) dt \vec{i} + \dots \vec{j} + \dots \vec{k} \quad \text{vector.}$$

position function $\vec{r}(t) \rightsquigarrow$ velocity $\vec{v}(t) \rightsquigarrow$ acceleration $\vec{a}(t) = \vec{v}'(t)$

$$\vec{r}(t) = \int_{t_0}^t \vec{v}(z) dz + \vec{r}(t_0) \leftarrow \boxed{\vec{v}(t) = \int_{t_0}^t \vec{a}(z) dz + \vec{v}(t_0)} \leftarrow \underline{a(t)}$$

$$\begin{cases} \vec{r}(t_0) = \vec{r}(t_0) \\ \vec{r}'(t) = \vec{v}(t) \end{cases}$$

$$\begin{cases} \vec{v}(t_0) = \vec{v}_0 \\ \vec{v}'(t) = \vec{a}(t) \end{cases}$$

$$\frac{d}{dt} \int_{t_0}^t f(z) dz = f(t).$$

Ex: $\vec{r}(0) = \langle -1, 2 \rangle, \quad \vec{v}(t) = 3t^2 \vec{i} + \cos(\pi t) \vec{j}$

$$\Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(z) dz = \langle -1, 2 \rangle + \int_0^t (3z^2 \vec{i} + \cos(\pi z) \vec{j}) dz$$

$$= \langle -1, 2 \rangle + \left[t^3 \vec{i} + \frac{1}{\pi} \sin(\pi t) \vec{j} \right]_0^t$$

$$= \langle -1, 2 \rangle + \left\langle t^3, \frac{1}{\pi} \sin(\pi t) \right\rangle = \langle -1 + t^3, 2 + \frac{1}{\pi} \sin(\pi t) \rangle$$

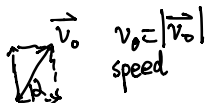
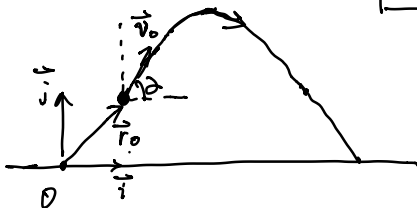
Application: projectile motion

$$g = 9.8 \text{ m/s}^2$$

$$\boxed{m \cdot \vec{a} = \vec{C} = -m \cdot g \cdot \vec{j}} \Rightarrow \underline{\vec{a} = -g \cdot \vec{j}}$$

$$\rightsquigarrow \vec{v}(t) = \vec{v}_0 + \int_0^t \underbrace{\vec{a}(z)}_{-g \cdot \vec{j}} dz = \vec{v}_0 - g t \cdot \vec{j}$$

$$\rightsquigarrow \vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(z) dz = \vec{r}_0 + \int_0^t (\vec{v}_0 - g z \vec{j}) dz = \vec{r}_0 + \vec{v}_0 \cdot t - \frac{1}{2} g t^2 \vec{j}$$



$$\langle x(t), y(t) \rangle = \langle x_0, y_0 \rangle + \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle \cdot t - \frac{1}{2} g t^2 \langle 0, 1 \rangle$$

$$= \left\langle \underbrace{x_0 + v_0 \cos \alpha \cdot t}_x, \underbrace{y_0 + v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2}_y \right\rangle, \quad t \geq 0.$$

$$t = \frac{x - x_0}{v_0 \cos \alpha} \rightarrow y - y_0 = v_0 \sin \alpha \cdot \frac{x - x_0}{v_0 \cos \alpha} - \frac{1}{2} g \left(\frac{x - x_0}{v_0 \cos \alpha} \right)^2$$

For simplicity, assume $(x_0, y_0) = (0, 0)$: $y = (\tan \alpha) \cdot x - \frac{g}{2 v_0^2 \cos^2 \alpha} x^2$ (a parabola)

• time of maximal height:

$$\vec{v}(t) = \vec{v}_0 - g t \cdot \vec{j} = \langle v_0 \cos \alpha, v_0 \sin \alpha - g t \rangle$$



$$t_1 = \frac{v_0 \sin \alpha}{g}$$

$$y_{\max} = v_0 \sin \alpha \cdot \frac{v_0 \sin \alpha}{g} - \frac{1}{2} \cdot g \cdot \frac{(v_0 \sin \alpha)^2}{g^2}$$

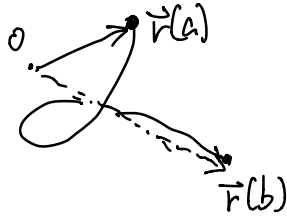
$$= \frac{v_0^2 \sin^2 \alpha}{g} - \frac{1}{2} \frac{v_0^2 \sin^2 \alpha}{g} = \frac{v_0^2 \sin^2 \alpha}{2g}$$

• Range

$$(v_0 \cos \alpha) \cdot t_1 \cdot 2 = v_0 \cos \alpha \cdot \frac{v_0 \sin \alpha}{g} \cdot 2$$

$$R = \frac{v_0^2 \sin(2\alpha)}{g}$$

13.3 Arc length



$$\langle x(t), y(t), z(t) \rangle = \vec{r}(t), \quad a \leq t \leq b$$

$$L = \int_a^b |\vec{r}'(t)| dt = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2} dt$$

EX: $\vec{r}(t) = \langle v_0 \cos \alpha \cdot t, v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2 \rangle$

$\alpha = 60^\circ = \frac{\pi}{3}$ $\cos \alpha = \frac{1}{2}$, $\sin \alpha = \frac{\sqrt{3}}{2}$, $v_0 = 1$, $g = 2$

$$= \langle \frac{1}{2}t, \frac{\sqrt{3}}{2}t - t^2 \rangle$$

$$\vec{r}'(t) = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} - 2t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2} - 2t\right)^2} = \frac{1}{2} \sqrt{1 + (\sqrt{3} - 4t)^2}$$

$$L = \int_0^{t_{\max}} |\vec{r}'(t)| dt = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{2} \sqrt{1 + (\sqrt{3} - 4t)^2} dt$$

$t_{\max} = \frac{\sqrt{3}}{2}$
 $\frac{\sqrt{3}}{2} t_{\max} - t_{\max}^2 = 0$

$$\left(x = \sqrt{3} - 4t, \quad t = \frac{\sqrt{3} - x}{4} \quad dt = -\frac{dx}{4} \right)$$

$$= \frac{1}{2} \int_{\sqrt{3}}^{\sqrt{3} - 4 \times \frac{\sqrt{3}}{2}} \sqrt{1 + x^2} \left(-\frac{dx}{4} \right)$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} = \int_{-\sqrt{3}}^0 + \int_0^{\sqrt{3}}$$

$$= -\frac{1}{2} \int_{\sqrt{3}}^{-\sqrt{3}} \sqrt{1+x^2} \frac{dx}{4} = \frac{1}{4} \int_0^{\sqrt{3}} \sqrt{1+x^2} dx$$

$$x = \tan \theta, \quad 1+x^2 = 1+\tan^2 \theta = \sec^2 \theta, \quad dx = d \frac{\sin \theta}{\cos \theta} = 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \sec^2 \theta$$

$$\frac{1}{2} \sqrt{\frac{3}{2}} \cdot \frac{1}{4} \int_0^{\frac{\pi}{3}} \sec \theta \cdot \sec^2 \theta d\theta = \frac{1}{4} \int_0^{\frac{\pi}{3}} \sec^3 \theta \cdot d\theta$$

$$\int \sqrt{1+x^2} dx = \left(\frac{1}{2} \sqrt{x^2+1} \cdot x + \sinh^{-1}(x) \right)$$

$$\frac{d\theta}{\cos^3 \theta} = \frac{\sin \theta d\theta}{\cos^4 \theta} = \frac{d \sin \theta}{(1-\sin^2 \theta)^2} = \frac{dy}{(1-y^2)^2}$$

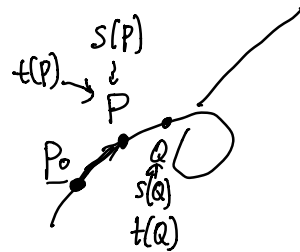
Ex: help: $\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$

$$\vec{r}'(t) = \langle -3 \sin(3t), 3 \cos(3t), 4 \rangle$$

$$|\vec{r}'(t)| = \sqrt{(-3)^2 \sin^2(3t) + 3^2 \cos^2(3t) + 4^2} = \sqrt{25} = 5$$

$$\int_a^b |\vec{r}'(t)| dt = \int_a^b 5 dt = 5 \cdot (b-a)$$

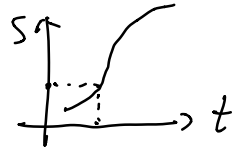
• Arclength parametrization $\vec{r} = \vec{r}(t)$



$$s = \int_{t_0}^t |\vec{r}'(t)| dt = s(t) \rightarrow \frac{ds}{dt} = |\vec{r}'(t)| > 0$$

s is an increasing function of t (since $\vec{r}(t)$ smooth $\rightarrow \vec{r}'(t) \neq 0$)

\leadsto solve t in terms of s $t = t(s)$



$\leadsto \vec{r}(t(s)) = \vec{r}(s)$ arclength parametrization.

Ex: $|\vec{r}'(t)| = 5$ let $t_0 = 0$

$$s = \int_0^t 5 dt = 5t \rightarrow t = \frac{s}{5}$$

$$\vec{r}(t(s)) = \left\langle \cos\left(3 \cdot \frac{s}{5}\right), \sin\left(3 \cdot \frac{s}{5}\right), 4 \cdot \frac{s}{5} \right\rangle$$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{3}{5}s\right), \sin\left(\frac{3}{5}s\right), \frac{4}{5}s \right\rangle}$$

arc length parametrization. (canonical intrinsic param.)

$$\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$$

$$\boxed{\vec{r}(u) = \langle \cos(3u^3), \sin(3u^3), 4u^3 \rangle}$$

$$t = u^3$$

$$\vec{r}(s) = \left\langle \cos\left(\frac{3}{5}s\right), \sin\left(\frac{3}{5}s\right), \frac{4}{5}s \right\rangle.$$

$$\vec{r} = \vec{r}(t) \rightsquigarrow \vec{r}(s) = \vec{r}(t(s)). \quad s \rightarrow t \rightarrow \vec{r}$$

$$\frac{d}{ds} \vec{r}(s) = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{1}{|\vec{r}'(t)|} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \text{ unit vector!}$$

$$s(t) = \int_{t_0}^t |\vec{r}'(z)| dz \quad \frac{ds}{dt} = |\vec{r}'(t)| \rightarrow \frac{dt}{ds} = \frac{1}{|\vec{r}'(t)|}$$

$$\frac{d\vec{r}}{ds} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} \quad \text{unit tangent vector at } t=t(s)$$

$$= \frac{\vec{r}'(t(s))}{|\vec{r}'(t(s))|} \quad \rightarrow \boxed{\left| \frac{d\vec{r}}{ds} \right| = 1}$$

Ex: $\vec{r}(t) = \langle \cos(3t), \sin(3t), 4t \rangle$

$$\vec{r}(s) = \left\langle \cos\left(\frac{3}{5}s\right), \sin\left(\frac{3}{5}s\right), \frac{4}{5}s \right\rangle$$

$$\frac{d\vec{r}}{ds} = \left\langle -\frac{3}{5} \sin\left(\frac{3}{5}s\right), \frac{3}{5} \cos\left(\frac{3}{5}s\right), \frac{4}{5} \right\rangle$$

$$\left| \frac{d\vec{r}}{ds} \right| = \sqrt{\left(\frac{3}{5}\right)^2 (\sin^2 + \cos^2) + \left(\frac{4}{5}\right)^2} = 1.$$