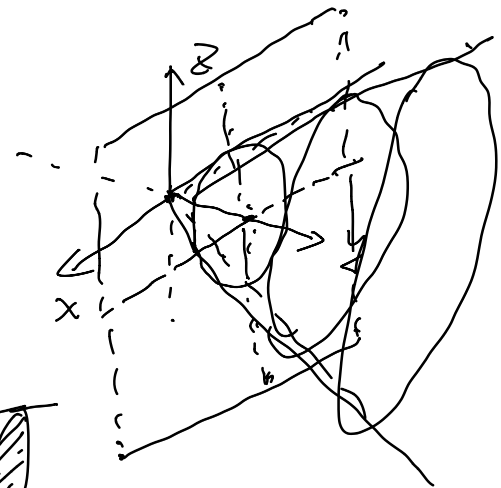
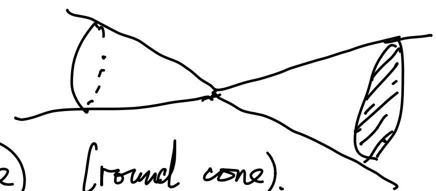
Ex: $x^2 - y^2 + z^2 = 0$

$$y^2 = x^2 + z^2$$

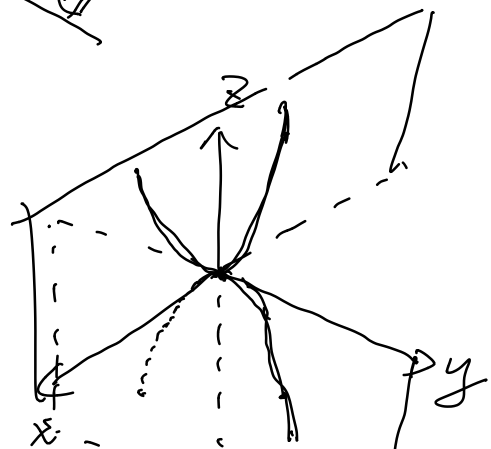


For fixed y, $x^2 + z^2 = y^2$ circle of radius $|y|$



elliptic cone (round cone).

$$\boxed{y^2 = \frac{x^2}{4} + \frac{z^2}{9}}$$



Ex: $z = \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2}}$

$y = 0$, $z = \frac{x^2}{a^2}$

$$\boxed{x = 0, \quad z = -\frac{y^2}{b^2}}$$

As y changes, $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

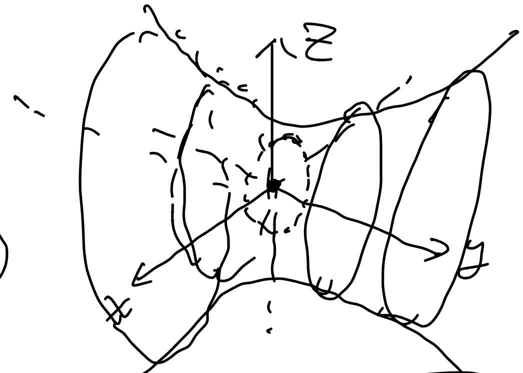
hyperbolic paraboloid:

$\boxed{\text{saddle surface}}$

Ex: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{y^2}{b^2} + 1$$

For Fix y, ellipse



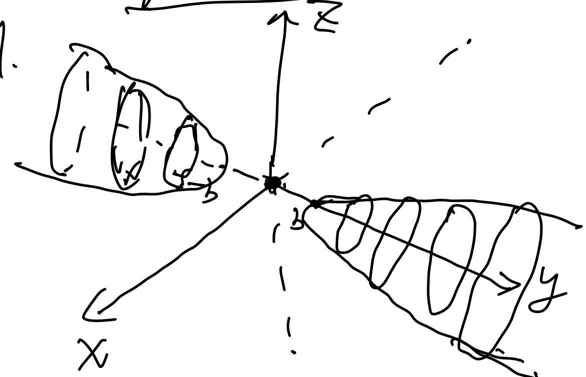
(one sheeted) hyperboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$$

$$\boxed{\frac{x^2}{a^2} + \frac{z^2}{c^2}} = \underbrace{\frac{y^2}{b^2} - 1}_{\geq 0}$$

$$\Downarrow$$

$$|y| \geq b$$



(two sheeted) hyperboloid.

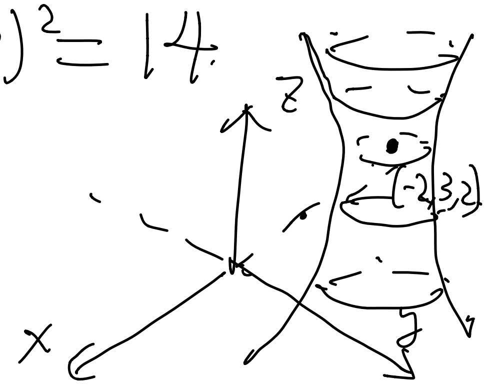
Ex: $2x^2 + y^2 - z^2 + 8x - 6y + 4z = 1.$

$$2(x^2 + 4x + 4) + (y^2 - 6y + 9) - (z^2 - 4z + 4)$$

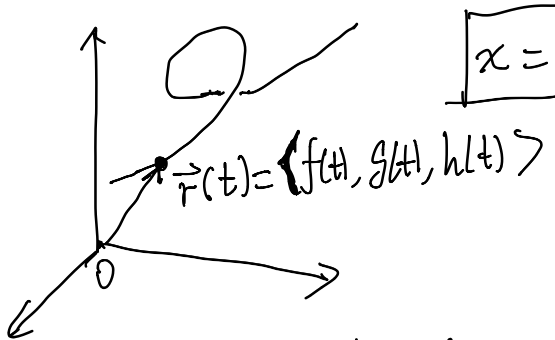
$$= 8 + 9 - 4 + 1 = 14$$

$$2(\underbrace{x-(-2)}_{u})^2 + (\underbrace{y-3}_{v})^2 - (\underbrace{z-2}_{w})^2 = 14.$$

$$2u^2 + v^2 - w^2 = 14.$$

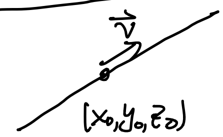


13.1: Curves in spaces



$\boxed{x = f(t)}$, $y = g(t)$, $z = h(t)$.

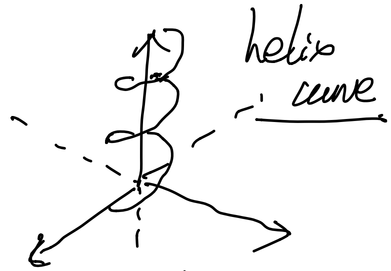
Ex:



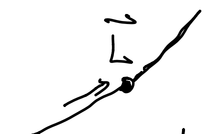
$x = x_0 + v_1 t$
$y = y_0 + v_2 t$
$z = z_0 + v_3 t$

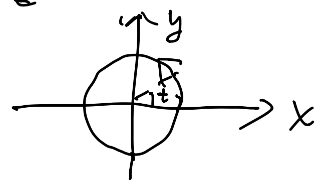
vector valued function: $\underset{\substack{\uparrow \\ \mathbb{R}}}{\overset{\substack{\text{time} \\ \downarrow}}{t}} \longmapsto \underset{\text{vector}}{\vec{r}(t)}$.

Ex: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

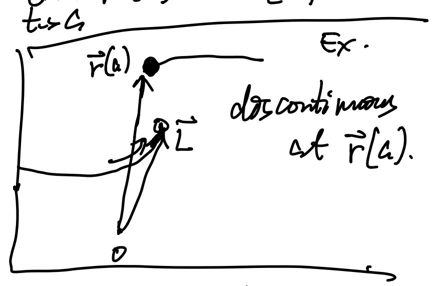


Def: · $\lim_{t \to a} \vec{r}(t) = \vec{L}$ if
$\lim_{t \to a} |\vec{r}(t) - \vec{L}| = 0$.





· $\vec{r}(t)$ is continuous at $a \in \mathbb{R}$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.



$\vec{r}(t)$ is continuous if it is continuous at any $a \in \mathbb{R}$.

· Derivative: $\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{1}{h} (\vec{r}(t+h) - \vec{r}(t))$.

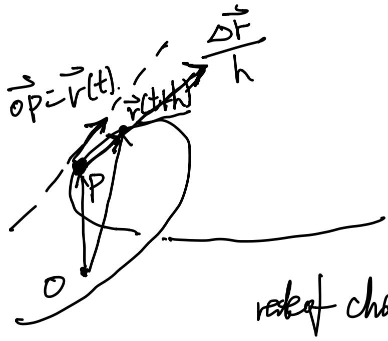
if $\vec{r}\,'(t)$ exists, then $\vec{r}(t)$ is differentiable at t.

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$.

$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

$\lim_{t\to a} \vec{r}(t)$ exists $\Leftrightarrow$ $\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$ all exists

$$\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle = \lim_{t\to a} \vec{r}(t).$$



$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}, \qquad \vec{r}(t+h) - \vec{r}(t) = \Delta \vec{r}(t).$$

a vector tangent to the curve at $\vec{r}(t)$.

rate of change of position vector

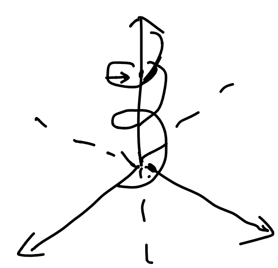
$=$

velocity $= \vec{v}(t)$. $\quad |\vec{v}(t)| =$ speed.

$\vec{v}(t) = |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}$ $\qquad \frac{\vec{v}}{|\vec{v}|}$ direction of velocity

$\vec{a}(t) = \frac{d}{dt} \vec{v}(t)$ $\quad$ acceleration = rate of change of the velocity

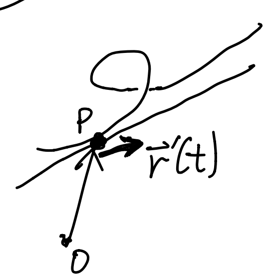
Ex: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.



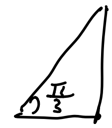
$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos t, -\sin t, 0 \rangle$

tangent line at $t = \frac{\pi}{3}$.



$$\vec{OP} = \vec{r}\left(\frac{\pi}{3}\right) = \left\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}, \frac{\pi}{3} \right\rangle$$



$$= \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right\rangle$$

$$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \right)$$

$$\vec{r}'\left(\frac{\pi}{3}\right) = \left\langle -\sin\frac{\pi}{3}, \cos\frac{\pi}{3}, 1 \right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 \right\rangle$$

tangent line: $\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}$

$$\left(\begin{array}{l} x(t) = \frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)t, \; y(t) = \frac{\sqrt{3}}{2} + \frac{1}{2}t, \\ z(t) = \frac{\pi}{3} + 1 \cdot t. \qquad -\infty < t < +\infty \end{array} \right)$$

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$$

speed at time t: $|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1^2}$

$$= \sqrt{2}$$

direction: $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$

Differentiation Rules: $\underset{}{\overset{\langle u_1,u_2,u_3\rangle}{\vec{r}(t)}}$, $\overset{\langle v_1,v_2,v_3\rangle}{\vec{s}(t)}$. 2 vector valued function

$$\frac{d}{dt}(\vec r+\vec s)=\frac{d}{dt}\vec r+\frac{d}{dt}\vec s$$

$$\frac{d}{dt}(f(t)\cdot\vec r)=f'\cdot\vec r+f\cdot\vec r\,'$$

$$\frac{d}{dt}(\vec r\bullet\vec s)=\vec r\,'\bullet\vec s+\vec r\bullet\vec s\,'$$

$\vec r\bullet\vec s=u_1v_1+u_2v_2+u_3v_3$

$$\boxed{\frac{d}{dt}(\vec r\times\vec s)=\vec r\,'\times\vec s+\vec r\times\vec s\,'}$$

$$\vec r\times\vec s=\begin{vmatrix}\vec i & \vec j & \vec k\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

Pf: $\frac{d}{dt}(\vec r\times\vec s)=\lim_{h\to 0}\frac{\vec r(t+h)\times\vec s(t+h)-\vec r(t)\times\vec s(t)}{h}$

$$=\lim_{h\to 0}\frac{\left(\vec r(t+h)\times\vec s(t+h)-\vec r(t)\times\vec s(t+h)\right)+\left(\vec r(t)\times\vec s(t+h)-\vec r(t)\times\vec s(t)\right)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\vec r(t+h)-\vec r(t)}{h}\right)\times\lim_{h\to 0}\left(\vec s(t+h)\right)+\lim_{h\to 0}\vec r(t)\times\lim_{h\to 0}\left(\frac{\vec s(t+h)-\vec s(t)}{h}\right).$$

$$=\vec r\,'(t)\times\vec s(t)+\vec r(t)\times\vec s\,'(t)=\frac{d}{dt}(\vec r\times\vec s).$$

- chain rule: $t\mapsto\vec r(t)$.

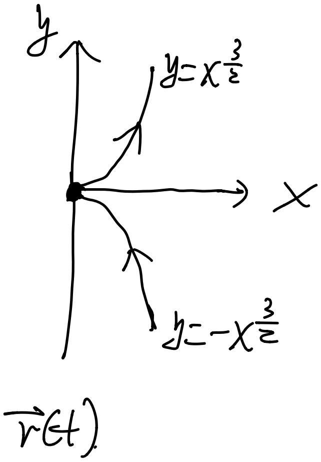
$$\begin{tikzcd} t \arrow[r, mapsto] \arrow[rr, bend right, "\vec r\circ f"'] & f(t) \arrow[r, mapsto] & \vec r(f(t)) \end{tikzcd}$$

$$\frac{d}{dt}\vec r(f(t))=\vec r\,'(f(t))\cdot f'(t).$$

Def: • $\vec{r}(t)$ is differentiable at t if $\vec{r}\,'(t)$ exists.

• $\vec{r}(t)$ is smooth if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$.

Ex: $\vec{r}(t) = \langle \underbrace{t^2}_{x}, \underbrace{t^3}_{y} \rangle$.



$$y = \left(x^{\frac{1}{2}}\right)^3 = x^{\frac{3}{2}}$$

$$y = -\left(x^{\frac{1}{2}}\right)^3 = -x^{\frac{3}{2}}$$

$\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle \Rightarrow$ differentiable

$\underline{\vec{r}\,'(0) = \vec{0}} \Longrightarrow$ $\underline{\vec{r}(t) \text{ is not smooth at } 0.}$

This is an example of <u>piecewise smooth curve</u>