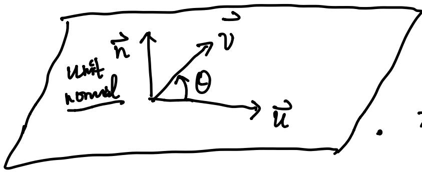
Cross Product

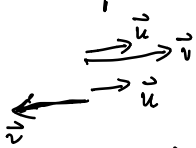


$$\vec{u}\times\vec{v} = \left(|\vec{u}||\vec{v}|\sin\theta\right)\cdot\vec{n} \qquad \vec{u}\times\vec{v} \perp \vec{u},\vec{v}$$

- $\vec{u}$ is parallel to $\vec{v}$ $\Leftrightarrow$ $\theta = 0$, or π $\Leftrightarrow$ $\vec{u}\times\vec{v} = \vec{0}$

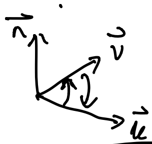
$$\theta = 0, \text{ or } \pi \;\Leftrightarrow\; \sin\theta = 0$$

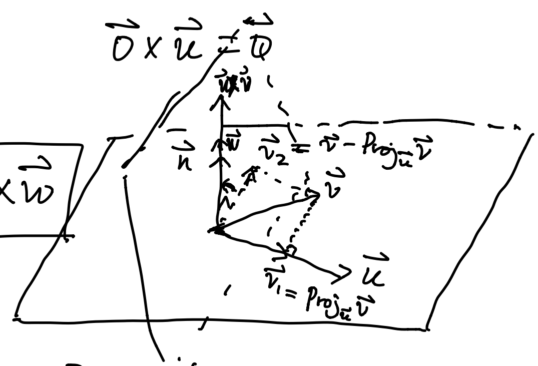
$$\vec{u}\perp\vec{v} \Leftrightarrow \vec{u}\bullet\vec{v} = 0$$



- properties: $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$. $\qquad r\vec{u}\times s\vec{v} = (rs)(\vec{u}\times\vec{v})$

$$\vec{0}\times\vec{u} = \vec{0}$$





$$\boxed{\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}}$$ ✓

step1: $\vec{u}\times\vec{v} = \vec{u}\times\vec{v}_2$

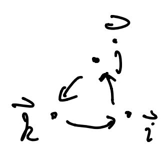
step2: $\vec{u}\times\vec{v}_2$ is the vector obtained in 2 more steps:

(2a). rotate $\vec{v}_2$ by 90° counterclockwise in the plane perpendicular to $\vec{u}$

(2b) rescale $\vec{w}$ by $|\vec{u}|$.

$$\boxed{(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\bullet\vec{w})\vec{v} - (\vec{v}\bullet\vec{w})\vec{u}}$$

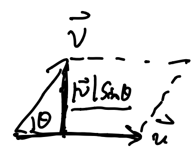
$$\vec{j}\times\vec{i} = -\vec{k}$$



$\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$

$\vec{i}\times\vec{j} = -\vec{j}\times\vec{i} = \vec{k}$, $\vec{j}\times\vec{k} = -\vec{k}\times\vec{j} = \vec{i}$, $\vec{k}\times\vec{i} = -\vec{i}\times\vec{k} = \vec{j}$

$\vec{i}\times\vec{i} = \vec{0} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k}$.



$\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\cdot\vec{n}$, $\quad |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ = area of parallelogram.

$$\vec{u}\bullet\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\vec{u}\bullet\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

$$\vec{u}\times\vec{v} = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$\overset{\text{distributive law}}{=} (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$\overset{\text{Laplace expansion}}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$(-1)^{1+2} = -1$

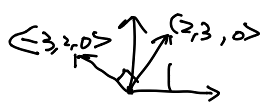
$(-1)^{i+j}$

$$\langle 0,0,1\rangle \times \langle 2,3,4\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \langle -3,\ 2,\ 0\rangle$$

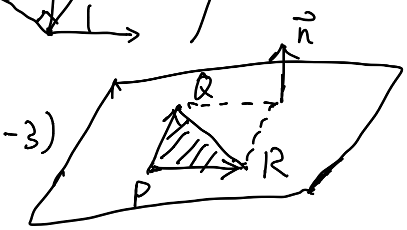
$$\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} \quad -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}$$

$$\left(\langle 0,0,1\rangle \times \underbrace{\langle 2,3,4\rangle}_{=\left(\langle 2,3,0\rangle + \langle 0,0,4\rangle\right)} = \langle 0,0,1\rangle \times \langle 2,3,0\rangle \right)$$

$=$



Ex: $P(1,2,3)$, $Q=(-1,0,1)$, $R(2,1,-3)$



- Calculate a unit normal perpendicular to plane

$\vec{PQ} = \langle -2,-2,-2\rangle$, $\vec{PR} = \langle 1,-1,-6\rangle$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & -2 \\ 1 & -1 & -6 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & -2 \\ -1 & -6 \end{vmatrix} - \vec{j}\cdot\begin{vmatrix} -2 & -2 \\ 1 & -6 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}\cdot\underbrace{\left((-2)\times(-6) - (-2)\times(-1)\right)}_{=10} - \vec{j}\cdot 14 + \vec{k}\cdot 4$$

unit normal $\quad \vec{v} = 10\vec{i} - 14\vec{j} + 4\vec{k}$

$$\vec{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{10\vec{i} - 14\vec{j} + 4\vec{k}}{\sqrt{10^2 + 14^2 + 4^2}}.$$

- Calculate the area of the triangle. $\quad |\vec{PQ}\times\vec{PR}| =$ area of parallelogram

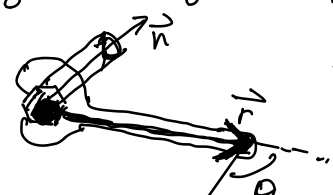
$$\text{Area of } \triangle_{PQR} = \frac{1}{2}\underline{|\vec{PQ}\times\vec{PR}|} = \frac{1}{2}\sqrt{\underbrace{10^2 + 14^2 + 4^2}_{=100+196+16}} = \frac{1}{2}\underbrace{\sqrt{312}}_{=\sqrt{78}}$$

296
16

$$\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\underline{\sin\theta}\cdot\vec{n}$$

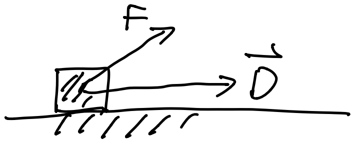
$$|u\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta.$$

- physical meaning: torque of a force.



torque vector $= \underline{\vec{r} \times \vec{F}} = (|\vec{r}||\vec{F}|\sin\theta) \cdot \vec{n}$

$\theta = 0, \pi$, $\underline{\vec{r} \times \vec{F} = \vec{0}}$



$\vec{F} \bullet \vec{D} = \underline{\mathrm{Proj}_{\vec{D}} \vec{F}} \bullet \vec{D}$ (work of $\vec{F}$ along $\vec{D}$)

$\uparrow$ displacement

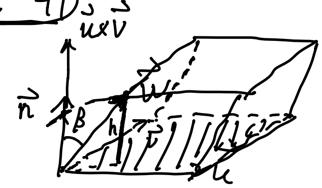
Triple scalar product of $\vec{u}, \vec{v}, \vec{w}$ (Box product)

$\underbrace{\underbrace{(\vec{u} \times \vec{v})}_{\text{vector}} \bullet \vec{w}}_{\text{scalar}}$, $|(\vec{u} \times \vec{v}) \bullet \vec{w}| = \underbrace{|\vec{u} \times \vec{v}|}_{\text{area of parallelogram}} |\vec{w}| |\cos\beta|$

$\parallel$

volume of the parallelopiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

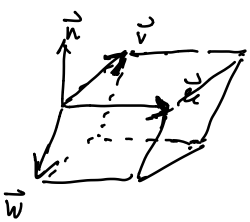
parallelopiped



$(\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \bullet (w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k})$

$= \left(\vec{i} \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$

$\bullet (w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k})$.



$\underline{\vec{u} \times \vec{v} \bullet \vec{w} < 0}$

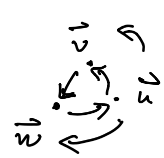
$\vec{i} \cdot \vec{i} = \vec{j} \bullet \vec{j} = \vec{k} \cdot \vec{k} = 1$.

$\vec{i} \bullet \vec{j} = 0 = \cdots$

$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} w_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} w_2 + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = - \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = -(\vec{v} \times \vec{u}) \bullet \vec{w}$

$= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (\vec{v} \times \vec{u}) \bullet \vec{w}$



$\vec{w} \times \vec{u} \bullet \vec{v}$

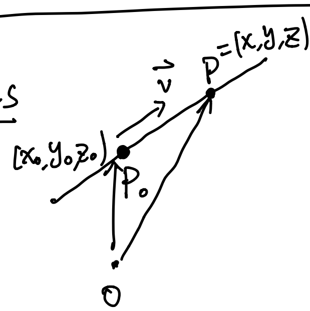
$\parallel$

$\vec{u} \times \vec{v} \bullet \vec{w}$

$\parallel$

$-(\vec{u} \times \vec{w} \bullet \vec{v})$

prove or disprove $(\vec{u}\times\vec{v})\times\vec{w} \overset{?}{=} \vec{u}\times(\vec{v}\times\vec{w})$ exercise.



Lines

$$\vec{OP}-\vec{OP_0}=\underbrace{\vec{P_0P}=t\cdot\vec{v}}_{t\text{: time parameter}}$$ ← velocity vector.

$$\vec{OP}=\vec{OP_0}+t\vec{v} \qquad -\infty<t<+\infty$$

$$\boxed{\vec{r}(t)=\vec{r}_0+t\vec{v}}$$ vector equation for the line L

$\Updownarrow$

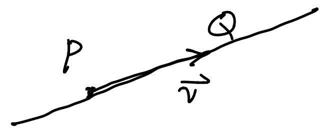
$$\langle x,y,z\rangle=\langle x_0,y_0,z_0\rangle+t\langle v_1,v_2,v_3\rangle$$

$\Updownarrow$

$$\boxed{x=x_0+tv_1,\quad y=y_0+tv_2,\quad z=z_0+tv_3}$$ standard parametrization of the line L

$-\infty<t<+\infty$

Ex: $P(1,2,3)$, $Q(3,2,1)$.



$$\vec{v}=\vec{OQ}-\vec{OP}=\langle 2,0,-2\rangle$$

Full line $x=1+2t$, $y=2+0t$, $z=3+(-2)t$ $\Longleftrightarrow$ $\vec{r}(t)=\underset{\vec{OP}}{\langle 1,2,3\rangle}+t\langle 2,0,-2\rangle$

$-\infty<t<+\infty$.

Line segment between P & Q:

$$\vec{r}(t)=\vec{OP}+t\cdot\vec{PQ} \quad 0\le t\le 1 \Longleftrightarrow x=1+2t,\ y=2,\ z=3-2t \quad 0\le t\le 1$$

$$\vec{r}(t)=\underset{\text{initial position}}{\vec{r}_0}+t\cdot\vec{v}=\vec{r}_0+t\,\underbrace{\underset{\text{speed}}{|\vec{v}|}\cdot\underset{\text{direction}}{\frac{\vec{v}}{|\vec{v}|}}}_{\text{velocity}}$$