$$(\vec{v}_1+\vec{v}_2)+\vec{v}_3 = \vec{v}_1+(\vec{v}_3+\vec{v}_2)$$

associativity $\parallel$ $\qquad$ $=$ commutativity

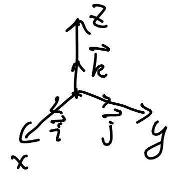
$$\vec{v}_1+(\vec{v}_2+\vec{v}_3)$$

$$k\cdot(\vec{v}_1+\vec{v}_2) = k\vec{v}_1 + k\vec{v}_2 \quad \text{distributivity.}$$

k: scalar $\qquad$ $(\vec{v}_1+\vec{v}_2)$: vector

$\boxed{\text{2-dim}}$ $\rightarrow$ $\vec{v} = \langle v_1, v_2\rangle$, $\vec{u} = \langle u_1, u_2\rangle$ $\qquad \vec{v}-\vec{u} = \langle v_1-u_1, v_2-u_2\rangle$

$\vec{v}$ is a Unit vector if $|\vec{v}|=1$.

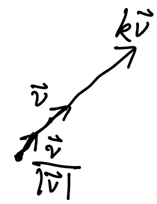


Ex: $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$

$$\vec{v} = \langle v_1, v_2, v_3\rangle = v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle$$

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$|k\vec{v}| = |k||\vec{v}|$$



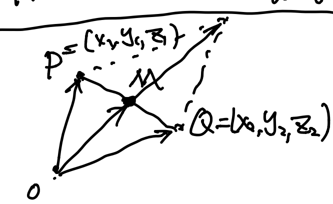
$\vec{0} \neq \vec{v}$ $\Leftrightarrow$ $|\vec{v}| \neq 0$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{|\vec{v}|}\cdot\vec{v} \quad \boxed{\text{unit vector}} \qquad \left|\frac{\vec{v}}{|\vec{v}|}\right| = \frac{1}{|\vec{v}|}\cdot|\vec{v}| = 1$$

$\frac{1}{|\vec{v}|}\cdot\vec{v}$: scalar multiplication

$\frac{\vec{v}}{|\vec{v}|}$: unit vector in the direction of $\vec{v}$.

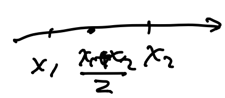
Ex:



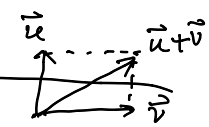
$$\vec{OM} = \frac{1}{2}\cdot(\vec{OP}+\vec{OQ}) = \frac{1}{2}\left(\langle x_1,y_1,z_1\rangle + \langle x_2,y_2,z_2\rangle\right)$$

$$= \frac{1}{2}\langle x_1+x_2, y_1+y_2, z_1+z_2\rangle$$

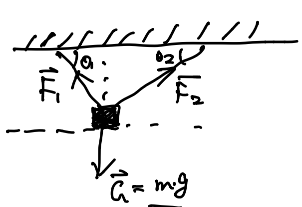
$$\Rightarrow M = \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2)\right).$$



Ex:



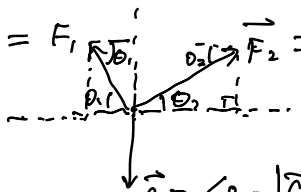
Ex:



Q: Given $\vec{G}$. Find $\vec{F}_1$ and $\vec{F}_2$.

Sol: $\vec{F}_1+\vec{F}_2+\vec{G} = 0$.

$\langle |\vec{F_1}|\cos\theta_1, |\vec{F_1}|\cdot\sin\theta_1\rangle = \vec{F_1}$



$\vec{F_2} = \langle |\vec{F_2}|\cos\theta_2, |\vec{F_2}|\sin\theta_2\rangle$

$\vec{G} = \langle 0, -|\vec{G}|\rangle$

$$0 = \vec{F_1}+\vec{F_2}+\vec{G} = \langle -|\vec{F_1}|\cos\theta_1 + |\vec{F_2}|\cos\theta_2, |\vec{F_1}|\sin\theta_1 + |\vec{F_2}|\sin\theta_2 - |\vec{G}|\rangle$$

$$\Updownarrow$$

$$\begin{cases} -|\vec{F_1}|\cos\theta_1 + |\vec{F_2}|\cos\theta_2 = 0 & ① \\ |\vec{F_1}|\sin\theta_1 + |\vec{F_2}|\sin\theta_2 = |\vec{G}| & ② \end{cases}$$

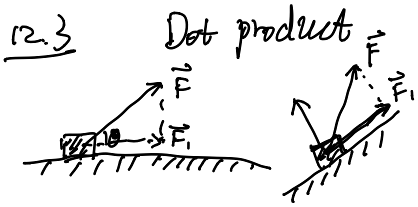
$$① \Rightarrow \underline{|\vec{F_2}| = |\vec{F_1}|\cdot\frac{\cos\theta_1}{\cos\theta_2}}$$

$$\Rightarrow |\vec{F_1}|\sin\theta_1 + |\vec{F_1}|\frac{\cos\theta_1}{\cos\theta_2}\sin\theta_2 = |\vec{G}|$$

$$|\vec{F_1}|\cdot\frac{1}{\cos\theta_2}\cdot\underbrace{(\sin\theta_1\cos\theta_2 + \cos\theta_1\cdot\sin\theta_2)}_{\sin(\theta_1+\theta_2)} = |\vec{F_1}|\frac{1}{\cos\theta_2}\sin(\theta_1+\theta_2)$$

$$\Rightarrow |\vec{F_1}| = |\vec{G}|\cdot\frac{\cos\theta_2}{\sin(\theta_1+\theta_2)}, \quad |\vec{F_2}| = |\vec{G}|\cdot\frac{\cos\theta_1}{\sin(\theta_1+\theta_2)}.$$

12.3 Dot product

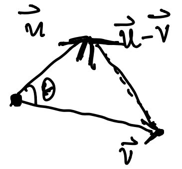


$\vec{u} = \langle u_1, u_2, u_3\rangle$

$\vec{v} = \langle v_1, v_2, v_3\rangle$

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3}$$

Dot product
$\parallel$
Scalar product.
$\parallel$
inner product

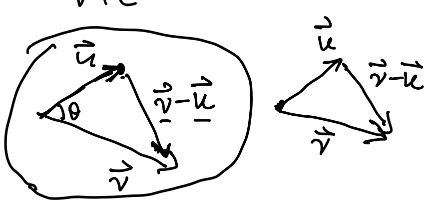


$\vec{v} + (\vec{u}-\vec{v}) = \vec{u}$

Law of cosine

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\cdot|\vec{v}|\cos\theta.$$

$$\Rightarrow \cos\theta = \frac{|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u}-\vec{v}|^2}{2|\vec{u}|\cdot|\vec{v}|}.$$



$$|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2$$

$$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

$$|\vec{u}-\vec{v}|^2 = (u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2$$

$$= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2$$

$$= |\vec{u}|^2 + |\vec{v}|^2 - 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}|\cdot|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

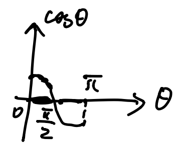
Dot product.

$0 \le \theta < \pi$

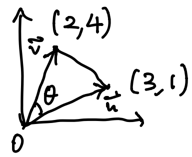
$0 \le \theta < \frac{\pi}{2}$ acute angle $\Leftrightarrow \cos\theta > 0. \Leftrightarrow \vec{u}\bullet\vec{v} > 0$

$\theta = \frac{\pi}{2}$ right angle $\Leftrightarrow \cos\theta = 0 \Leftrightarrow \vec{u}\bullet\vec{v} = 0$

$\frac{\pi}{2} < \theta < \pi$ obtuse angle $\Leftrightarrow \cos\theta < 0 \Leftrightarrow \vec{u}\bullet\vec{v} < 0$



Ex.



$\vec{u} = \langle 3,1\rangle$, $\vec{v} = \langle 2,4\rangle$ $\quad \vec{u}\bullet\vec{v} = 3\times 2 + 1\times 4 = 10$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \frac{10}{\sqrt{3^2+1^2}\cdot\sqrt{2^2+4^2}} = \frac{10}{\sqrt{10}\cdot\sqrt{20}} = \frac{10}{10\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ = \frac{\pi}{4}.$$

Properties: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ $\qquad \vec{u}\cdot(k\vec{v}) = k(\vec{u}\cdot\vec{v})$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$ $\qquad k$ ↑ scalar

$\vec{u}\cdot\vec{u} = |\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2.$ $\qquad \vec{0}\cdot\vec{u} = \cancel{\vec{0}}$? 0 $\qquad \vec{u}\cdot\vec{v}$ scalar.

$\boxed{0\cdot\vec{u} = \vec{0}}$ ✓ $\qquad k\cdot\vec{u}$ vector

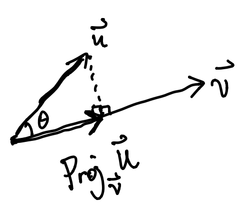
$0 = \underbrace{\vec{0}\bullet\vec{u}}_{\text{scalar}} \ne \underbrace{\vec{0}}_{\text{vector}} = \langle 0,0,0\rangle$ $\qquad$ scalar $\neq$ vector

scalar multiplication

Recall: $\vec{v} = |\vec{v}|\cdot\frac{\vec{v}}{|\vec{v}|}$

unit vector representing direction of $\vec{v}$

Projection formula



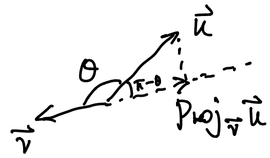
$$\text{Proj}_{\vec{v}}\vec{u} = (|\vec{u}|\cdot\cos\theta)\frac{\vec{v}}{|\vec{v}|}$$

$$= |\text{Proj}_{\vec{v}}\vec{u}|\cdot(\text{direction of } \text{Proj}_{\vec{v}}\vec{u})$$

$$= |\vec{u}|\cdot\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|} = \underbrace{\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\right)}_{\text{scalar}}\vec{v} = \left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

$$\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|}\right) = (|\vec{u}|\cos\theta)\cdot\frac{\vec{v}}{|\vec{v}|}$$

component of $\vec{u}$ in the direction of $\vec{v}$.

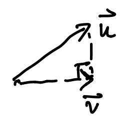


$$\mathrm{Proj}_{\vec{v}}\vec{u} = \left(|\vec{u}|\cdot\cos(\pi-\theta)\right)\cdot\left(-\frac{\vec{v}}{|\vec{v}|}\right)$$
$$= |\vec{u}|\cdot(-\cos\theta)\cdot\left(-\frac{\vec{v}}{|\vec{v}|}\right) = |\vec{u}|\cos\theta\cdot\frac{\vec{v}}{|\vec{v}|}$$
$$= \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}\ \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}$$

comp. of $\vec{u}$ along $\vec{v}$ = directed length of the projection

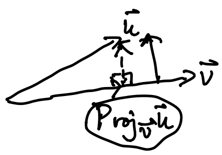
Ex: $\vec{u} = \langle 1,2,3\rangle$ $\qquad \mathrm{Proj}_{\vec{v}}\vec{u} = \dfrac{1\times(-1)+2\times 0+3\times 2}{(-1)^2+0^2+2^2}\cdot\langle -1,0,2\rangle$

$\vec{v} = \langle -1,0,2\rangle$ $\qquad = \dfrac{-1+6}{1+4}\cdot\langle -1,0,2\rangle = \dfrac{5}{5}\cdot\langle -1,0,2\rangle = \underbrace{\langle -1,0,2\rangle}_{=\vec{v}}$ ✓



$\vec{u}-\vec{v} = \langle 2,2,1\rangle \perp \vec{v}$? ✓ $\quad (\vec{u}-\vec{v})\cdot\vec{v} = 2\times(-1)+2\times 0+1\times 2 = 0.$

Fact:



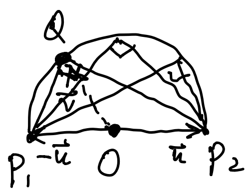
$\vec{u} = \underline{\mathrm{Proj}_{\vec{v}}\vec{u}} + \underline{(\vec{u}-\mathrm{Proj}_{\vec{v}}\vec{u})}$ orthogonal to $\vec{v}$.

$(\vec{u}-\mathrm{Proj}_{\vec{v}}\vec{u}) \perp \vec{v}$. ✓

Proof: $(\vec{u}-\mathrm{Proj}_{\vec{v}}\vec{u})\cdot\vec{v} = \vec{u}\cdot\vec{v} - \underline{\mathrm{Proj}_{\vec{v}}\vec{u}\cdot\vec{v}} = 0.$

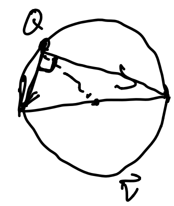
$$\mathrm{Proj}_{\vec{v}}\vec{u}\cdot\vec{v} = \left(\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right)\cdot\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{v}|^2}}\cancel{(\vec{v}\cdot\vec{v})} = \vec{u}\cdot\vec{v}$$

Ex:
Application



$P_1Q \perp P_2Q$?

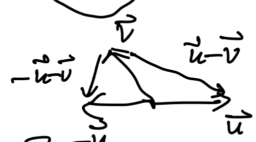
$\Updownarrow$

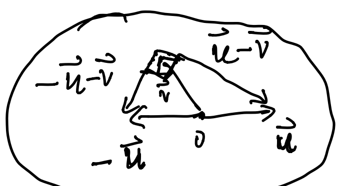


$\boxed{(-\vec{u}-\vec{v}) \perp (\vec{u}-\vec{v})}$?

$\Updownarrow$

$\boxed{(-\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = 0}$?



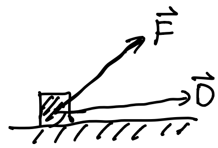


$\vec{u}\cdot(\vec{v}+\vec{w})$
$\quad \shortparallel$
$\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

$\underline{(-\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} = -\vec{u}\cdot\vec{u} + \cancel{\underline{\vec{u}\cdot\vec{v}}} - \cancel{\underline{\vec{v}\cdot\vec{u}}} + \vec{v}\cdot\vec{v}$

$= \underline{-|\vec{u}|^2 + |\vec{v}|^2} = 0$ ✓

Ex:



$$\text{Proj}_{\vec{D}}\vec{F}\cdot\vec{D} = \frac{\vec{F}\cdot\vec{D}}{|\vec{D}|^2}\vec{D}\cdot\vec{D} = \vec{F}\cdot\vec{D}$$