B

Appendix

The goal of this appendix is to establish the notation, terminology, and algebraic skills that are essential when learning calculus.

Algebra

EXAMPLE 1 Algebra review

a. Evaluate $(-32)^{2/5}$.

b. Simplify $\dfrac{1}{x-2} - \dfrac{1}{x+2}$.

c. Solve the equation $\dfrac{x^4 - 5x^2 + 4}{x - 1} = 0$.

SOLUTION

a. Recall that $(-32)^{2/5} = ((-32)^{1/5})^2$. Because $(-32)^{1/5} = \sqrt[5]{-32} = -2$, we have $(-32)^{2/5} = (-2)^2 = 4$.
Another option is to write $(-32)^{2/5} = ((-32)^2)^{1/5} = 1024^{1/5} = 4$.

b. Finding a common denominator and simplifying leads to

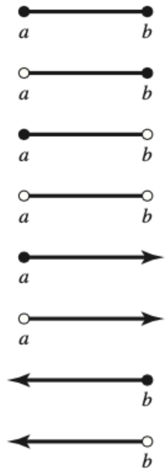
$$\frac{1}{x-2} - \frac{1}{x+2} = \frac{(x+2)-(x-2)}{(x-2)(x+2)} = \frac{4}{x^2-4}.$$

c. Notice that $x = 1$ cannot be a solution of the equation because the left side of the equation is undefined at $x = 1$. Because $x - 1 \neq 0$, both sides of the equation can be multiplied by $x - 1$ to produce $x^4 - 5x^2 + 4 = 0$. After factoring, this equation becomes $(x^2 - 4)(x^2 - 1) = 0$, which implies $x^2 - 4 = (x-2)(x+2) = 0$ or $x^2 - 1 = (x-1)(x+1) = 0$. The roots of $x^2 - 4 = 0$ are $x = \pm 2$, and the roots of $x^2 - 1 = 0$ are $x = \pm 1$. Excluding $x = 1$, the roots of the original equation are $x = -1$ and $x = \pm 2$.

Related Exercises 15, 20, 22 ◄

Sets of Real Numbers

Figure B.1 shows the notation for **open intervals**, **closed intervals**, and various **bounded** and **unbounded intervals**. Notice that either interval notation or set notation may be used.



$[a, b] = \{x: a \le x \le b\}$	Closed, bounded interval
$(a, b] = \{x: a < x \le b\}$	Bounded interval
$[a, b) = \{x: a \le x < b\}$	Bounded interval
$(a, b) = \{x: a < x < b\}$	Open, bounded interval
$[a, \infty) = \{x: x \ge a\}$	Unbounded interval
$(a, \infty) = \{x: x > a\}$	Unbounded interval
$(-\infty, b] = \{x: x \le b\}$	Unbounded interval
$(-\infty, b) = \{x: x < b\}$	Unbounded interval
$(-\infty, \infty) = \{x: -\infty < x < \infty\}$	Unbounded interval

Figure B.1

EXAMPLE 2 **Solving inequalities** Solve the following inequalities.

a. $-x^2 + 5x - 6 < 0$ **b.** $\dfrac{x^2 - x - 2}{x - 3} \le 0$

SOLUTION

a. We multiply by -1, reverse the inequality, and then factor:

$$x^2 - 5x + 6 > 0 \quad \text{Multiply by } -1.$$
$$(x - 2)(x - 3) > 0. \quad \text{Factor.}$$

The roots of the corresponding equation $(x - 2)(x - 3) = 0$ are $x = 2$ and $x = 3$. These roots partition the number line (Figure B.2) into three intervals: $(-\infty, 2)$, $(2, 3)$, and $(3, \infty)$. On each interval, the product $(x - 2)(x - 3)$ does not change sign. To determine the sign of the product on a given interval, a **test value** x is selected and the sign of $(x - 2)(x - 3)$ is determined at x.

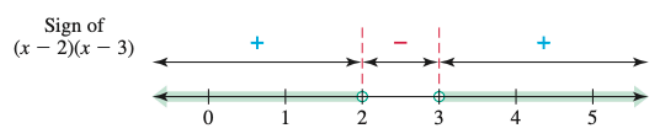


Figure B.2

A convenient choice for x in $(-\infty, 2)$ is $x = 0$. At this test value,

$$(x - 2)(x - 3) = (-2)(-3) > 0.$$

Using a test value of $x = 2.5$ in the interval $(2, 3)$, we have

$$(x - 2)(x - 3) = (0.5)(-0.5) < 0.$$

A test value of $x = 4$ in $(3, \infty)$ gives

$$(x - 2)(x - 3) = (2)(1) > 0.$$

Therefore, $(x - 2)(x - 3) > 0$ on $(-\infty, 2)$ and $(3, \infty)$. We conclude that the inequality $-x^2 + 5x - 6 < 0$ is satisfied for all x in either $(-\infty, 2)$ or $(3, \infty)$ (Figure B.2).

➤ The set of numbers $\{x: x \text{ is in } (-\infty, 2) \text{ or } (3, \infty)\}$ may also be expressed using the union symbol:

$$(-\infty, 2) \cup (3, \infty).$$

b. The expression $\frac{x^2 - x - 2}{x - 3}$ can change sign only at points where the numerator or denominator of $\frac{x^2 - x - 2}{x - 3}$ equals 0. Because

$$\frac{x^2 - x - 2}{x - 3} = \frac{(x + 1)(x - 2)}{x - 3},$$

the numerator is 0 when $x = -1$ or $x = 2$, and the denominator is 0 at $x = 3$. Therefore, we examine the sign of $\frac{(x + 1)(x - 2)}{x - 3}$ on the intervals $(-\infty, -1)$, $(-1, 2)$, $(2, 3)$, and $(3, \infty)$.

Using test values on these intervals, we see that $\frac{(x + 1)(x - 2)}{x - 3} < 0$ on $(-\infty, -1)$ and $(2, 3)$. Furthermore, the expression is 0 when $x = -1$ and $x = 2$. Therefore, $\frac{x^2 - x - 2}{x - 3} \le 0$ for all values of x in either $(-\infty, -1]$ or $[2, 3)$. (**Figure B.3**).

Test Value	$x + 1$	$x - 2$	$x - 3$	Result
-2	$-$	$-$	$-$	$-$
0	$+$	$-$	$-$	$+$
2.5	$+$	$+$	$-$	$-$
4	$+$	$+$	$+$	$+$

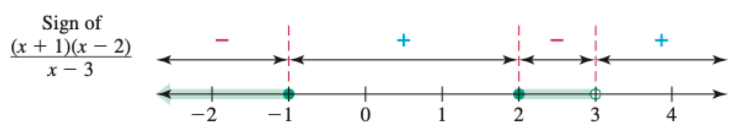


Figure B.3

Related Exercises 27, 29 ◄

Absolute Value

The **absolute value** of a real number x, denoted $|x|$, is the distance between x and the origin on the number line (**Figure B.4**). More generally, $|x - y|$ is the distance between the points x and y on the number line. The absolute value has the following definition and properties.

➤ The absolute value is useful in simplifying square roots. Because $\sqrt{a}$ is nonnegative, we have $\sqrt{a^2} = |a|$. For example, $\sqrt{3^2} = 3$ and $\sqrt{(-3)^2} = \sqrt{9} = 3$. Note that the solutions of $x^2 = 9$ are $|x| = 3$ or $x = \pm 3$.

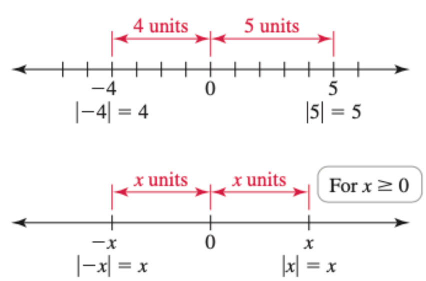


Figure B.4

Definition and Properties of the Absolute Value

The absolute value of a real number x is defined as

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Let a be a positive real number.

1. $|x| = a \Leftrightarrow x = \pm a$

2. $|x| < a \Leftrightarrow -a < x < a$

3. $|x| > a \Leftrightarrow x > a$ or $x < -a$

4. $|x| \le a \Leftrightarrow -a \le x \le a$

5. $|x| \ge a \Leftrightarrow x \ge a$ or $x \le -a$

6. $|x + y| \le |x| + |y|$

➤ Property 6 is called the **triangle inequality**.

EXAMPLE 3 **Inequalities with absolute values** Solve the following inequalities. Then sketch the solution on the number line and express it in interval notation.

a. $|x - 2| < 3$ **b.** $|2x - 6| \geq 10$

SOLUTION

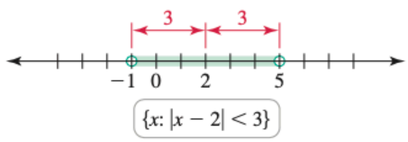


Figure B.5

a. Using property 2 of the absolute value, $|x - 2| < 3$ is written as

$$-3 < x - 2 < 3.$$

Adding 2 to each term of these inequalities results in $-1 < x < 5$ (Figure B.5). This set of numbers is written as $(-1, 5)$ in interval notation.

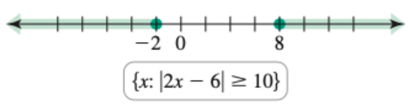


Figure B.6

b. Using property 5, the inequality $|2x - 6| \geq 10$ implies that

$$2x - 6 \geq 10 \quad \text{or} \quad 2x - 6 \leq -10.$$

We add 6 to both sides of the first inequality to obtain $2x \geq 16$, which implies $x \geq 8$. Similarly, the second inequality yields $x \leq -2$ (Figure B.6). In interval notation, the solution is $(-\infty, -2]$ or $[8, \infty)$.

Related Exercise 31 ◄

Cartesian Coordinate System

The conventions of the **Cartesian coordinate system** or ***xy*-coordinate system** are illustrated in Figure B.7. The set of real numbers is often denoted $\mathbb{R}$. The set of all ordered pairs of real numbers, which constitute the *xy*-plane, is often denoted $\mathbb{R}^2$.

➤ The familiar (x, y) coordinate system is named after René Descartes (1596–1650). However, it was introduced independently and simultaneously by Pierre de Fermat (1601–1665).

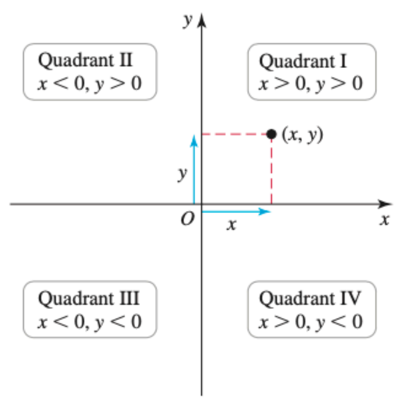


Figure B.7

Distance Formula and Circles

By the Pythagorean theorem (Figure B.8), we have the following formula for the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

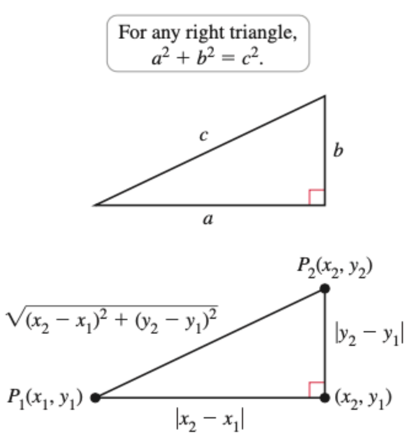


Figure B.8

> **Distance Formula**
>
> The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
>
> $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

A **circle** is the set of points in the plane whose distance from a fixed point (the **center**) is constant (the **radius**). This definition leads to the following equations that describe a circle.

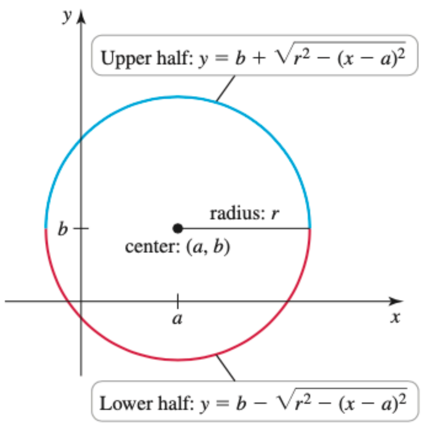


Figure B.9

Equation of a Circle

The equation of a circle centered at (a, b) with radius r is

$$(x - a)^2 + (y - b)^2 = r^2.$$

Solving for y, the equations of the upper and lower halves of the circle (**Figure B.9**) are

$$y = b + \sqrt{r^2 - (x - a)^2} \quad \text{Upper half of the circle}$$
$$y = b - \sqrt{r^2 - (x - a)^2}. \quad \text{Lower half of the circle}$$

EXAMPLE 4 **Sets involving circles**

a. Find the equation of the circle with center $(2, 4)$ passing through $(-2, 1)$.

b. Describe the set of points satisfying $x^2 + y^2 - 4x - 6y < 12$.

SOLUTION

a. The radius of the circle equals the length of the line segment between the center $(2, 4)$ and the point on the circle $(-2, 1)$, which is

$$\sqrt{(2 - (-2))^2 + (4 - 1)^2} = 5.$$

Therefore, the equation of the circle is

$$(x - 2)^2 + (y - 4)^2 = 25.$$

b. To put this inequality in a recognizable form, we complete the square on the left side of the inequality:

$$\begin{aligned} x^2 + y^2 - 4x - 6y &= x^2 - 4x \underbrace{+ 4 - 4}_{\substack{\text{Add and subtract the square} \\ \text{of half the coefficient of } x.}} + y^2 - 6y \underbrace{+ 9 - 9}_{\substack{\text{Add and subtract the square} \\ \text{of half the coefficient of } y.}} \\ &= \underbrace{x^2 - 4x + 4}_{(x-2)^2} + \underbrace{y^2 - 6y + 9}_{(y-3)^2} - 4 - 9 \\ &= (x - 2)^2 + (y - 3)^2 - 13. \end{aligned}$$

➤ Recall that the procedure shown here for completing the square works when the coefficient on the quadratic term is 1. When the coefficient is not 1, it must be factored out before completing the square.

Therefore, the original inequality becomes

$$(x - 2)^2 + (y - 3)^2 - 13 < 12, \quad \text{or} \quad (x - 2)^2 + (y - 3)^2 < 25.$$

This inequality describes those points that lie within the circle centered at $(2, 3)$ with radius 5 (**Figure B.10**). Note that a dashed curve is used to indicate that the circle itself is not part of the solution.

➤ A **circle** is the set of all points whose distance from a fixed point is a constant. A **disk** is the set of all points within and possibly on a circle.

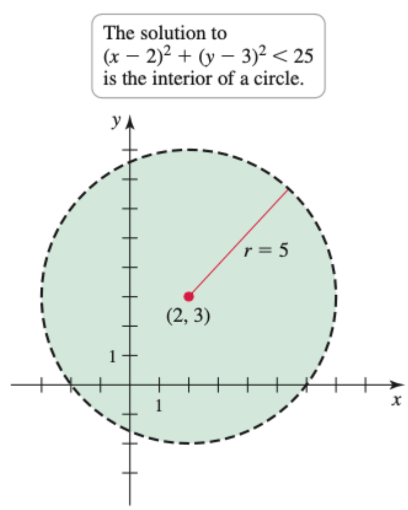


Figure B.10

Related Exercises 35–36 ◄

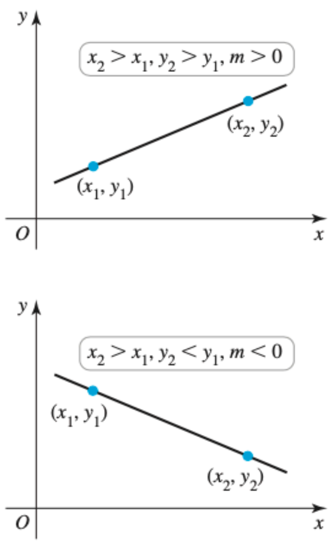


Figure B.11

Equations of Lines

The **slope** m of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the *rise over run* (**Figure B.11**), computed as

$$m = \frac{\text{change in vertical coordinate}}{\text{change in horizontal coordinate}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

> **Equations of a Line**
>
> **Point-slope form** The equation of the line with slope m passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.
>
> **Slope-intercept form** The equation of the line with slope m and y-intercept $(0, b)$ is $y = mx + b$ (**Figure B.12a**).
>
> **General linear equation** The equation $Ax + By + C = 0$ describes a line in the plane, provided A and B are not both zero.
>
> **Vertical and horizontal lines** The vertical line that passes through $(a, 0)$ has an equation $x = a$; its slope is undefined. The horizontal line through $(0, b)$ has an equation $y = b$, with slope equal to 0 (**Figure B.12b**).

➤ Given a particular line, we often talk about *the* equation of a line. But the equation of a specific line is not unique. Having found one equation, we can multiply it by any nonzero constant to produce another equation of the same line.

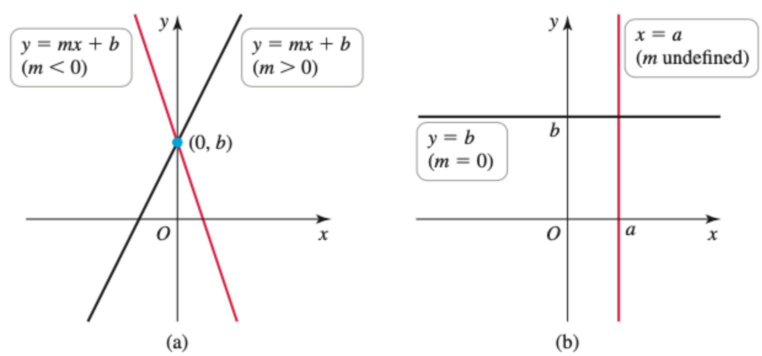


Figure B.12

EXAMPLE 5 **Working with linear equations** Find an equation of the line passing through the points $(1, -2)$ and $(-4, 5)$.

SOLUTION The slope of the line through the points $(1, -2)$ and $(-4, 5)$ is

$$m = \frac{5 - (-2)}{-4 - 1} = \frac{7}{-5} = -\frac{7}{5}.$$

Using the point $(1, -2)$, the point-slope form of the equation is

$$y - (-2) = -\frac{7}{5}(x - 1).$$

Solving for y yields the slope-intercept form of the equation:

$$y = -\frac{7}{5}x - \frac{3}{5}.$$

Related Exercise 39 ◄

➤ Because both points $(1, -2)$ and $(-4, 5)$ lie on the line and must satisfy the equation of the line, either point can be used to determine an equation of the line.

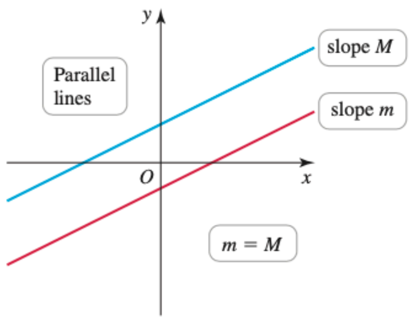


Figure B.13

Parallel and Perpendicular Lines

Two lines in the plane may have either of two special relationships to each other: They may be parallel or perpendicular.

> **Parallel Lines**
>
> Two distinct nonvertical lines are **parallel** if they have the same slope; that is, the lines with equations $y = mx + b$ and $y = Mx + B$ are parallel if and only if $m = M$ (Figure B.13). Two distinct vertical lines are parallel.

EXAMPLE 6 **Parallel lines** Find an equation of the line parallel to $3x - 6y + 12 = 0$ that intersects the x-axis at $(4, 0)$.

SOLUTION

Solving the equation $3x - 6y + 12 = 0$ for y, we have

$$y = \frac{1}{2}x + 2.$$

This line has a slope of $\frac{1}{2}$ and any line parallel to it has a slope of $\frac{1}{2}$. Therefore, the line that passes through $(4, 0)$ with slope $\frac{1}{2}$ has the point-slope equation $y - 0 = \frac{1}{2}(x - 4)$. After simplifying, an equation of the line is

$$y = \frac{1}{2}x - 2.$$

Notice that the slopes of the two lines are the same; only the y-intercepts differ.

Related Exercise 42 ◀

➤ The slopes of perpendicular lines are *negative reciprocals* of each other.

> **Perpendicular Lines**
>
> Two lines with slopes $m \neq 0$ and $M \neq 0$ are **perpendicular** if and only if $mM = -1$, or equivalently, $m = -1/M$ (Figure B.14).

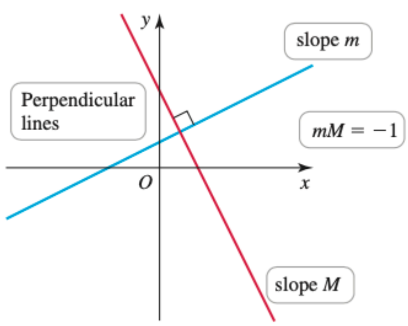


Figure B.14

EXAMPLE 7 **Perpendicular lines** Find an equation of the line passing through the point $(-2, 5)$ perpendicular to the line ℓ: $4x - 2y + 7 = 0$.

SOLUTION

The equation of ℓ can be written $y = 2x + \frac{7}{2}$, which reveals that its slope is 2. Therefore, the slope of any line perpendicular to ℓ is $-\frac{1}{2}$. The line with slope $-\frac{1}{2}$ passing through the point $(-2, 5)$ is

$$y - 5 = -\frac{1}{2}(x + 2), \quad \text{or} \quad y = -\frac{x}{2} + 4.$$

Related Exercise 43 ◀

APPENDIX B EXERCISES

Getting Started

1. State the meaning of $\{x: -4 < x \leq 10\}$. Express the set $\{x: -4 < x \leq 10\}$ using interval notation and draw it on a number line.
2. Write the interval $(-\infty, 2)$ in set notation and draw it on a number line.
3. Give the definition of $|x|$.
4. Write the inequality $|x - 2| \leq 3$ without absolute value symbols.
5. Write the inequality $|2x - 4| \geq 3$ without absolute value symbols.
6. Write an equation of the set of all points that are a distance of 5 units from the point $(2, 3)$.
7. Explain how to find the distance between two points whose coordinates are known.
8. Sketch the set of points $\{(x, y): x^2 + (y - 2)^2 > 16\}$.

9. Give an equation of the upper half of the circle centered at the origin with radius 6.

10. What are the possible solution sets of the equation $x^2 + y^2 + Cx + Dy + E = 0$?

11. Give an equation of the line with slope m that passes through the point $(4, -2)$.

12. Give an equation of the line with slope m and y-intercept $(0, 6)$.

13. What is the relationship between the slopes of two parallel lines?

14. What is the relationship between the slopes of two perpendicular lines?

Practice Exercises

15–20. Algebra review *Simplify or evaluate the following expressions without a calculator.*

15. $(1/8)^{-2/3}$

16. $\sqrt[3]{-125} + \sqrt{1/25}$

17. $(u + v)^2 - (u - v)^2$

18. $\dfrac{(a + h)^2 - a^2}{h}$

19. $\dfrac{1}{x + h} - \dfrac{1}{x}$

20. $\dfrac{2}{x + 3} - \dfrac{2}{x - 3}$

21–26. Algebra review

21. Factor $y^2 - y^{-2}$.

22. Solve $x^3 - 9x = 0$.

23. Solve $u^4 - 11u^2 + 18 = 0$.

24. Solve $4^x - 6(2^x) = -8$.

25. Simplify $\dfrac{(x + h)^3 - x^3}{h}$, for $h \neq 0$.

26. Rewrite $\dfrac{\sqrt{x + h} - \sqrt{x}}{h}$, where $h \neq 0$, without square roots in the numerator.

27–30. Solving inequalities *Solve the following inequalities and draw the solution on a number line.*

27. $x^2 - 6x + 5 < 0$

28. $\dfrac{x + 1}{x + 2} < 6$

29. $\dfrac{x^2 - 9x + 20}{x - 6} \leq 0$

30. $x\sqrt{x - 1} > 0$

31–34. Inequalities with absolute values *Solve the following inequalities. Then draw the solution on a number line and express it using interval notation.*

31. $|3x - 4| > 8$

32. $1 \leq |x| \leq 10$

33. $3 < |2x - 1| < 5$

34. $2 < |\frac{x}{2} - 5| < 6$

35–36. Circle calculations *Solve the following problems.*

35. Find the equation of the lower half of the circle with center $(-1, 2)$ and radius 3.

36. Describe the set of points that satisfy $x^2 + y^2 + 6x + 8y \geq 25$.

37–40. Working with linear equations *Find an equation of the line ℓ that satisfies the given condition. Then draw the graph of ℓ.*

37. ℓ has slope $5/3$ and y-intercept $(0, 4)$.

38. ℓ has undefined slope and passes through $(0, 5)$.

39. ℓ has y-intercept $(0, -4)$ and x-intercept $(5, 0)$.

40. ℓ is parallel to the x-axis and passes through the point $(2, 3)$.

41–42. Parallel lines *Find an equation of the following lines and draw their graphs.*

41. The line with y-intercept $(0, 12)$ parallel to the line $x + 2y = 8$

42. The line with x-intercept $(-6, 0)$ parallel to the line $2x - 5 = 0$

43–44. Perpendicular lines *Find an equation of the following lines.*

43. The line passing through $(3, -6)$ perpendicular to the line $y = -3x + 2$

44. The perpendicular bisector of the line segment joining the points $(-9, 2)$ and $(3, -5)$

Explorations and Challenges

45. **Explain why or why not** State whether the following statements are true and give an explanation or counterexample.
    - **a.** $\sqrt{16} = \pm 4$.
    - **b.** $\sqrt{4^2} = \sqrt{(-4)^2}$.
    - **c.** There are two real numbers that satisfy the condition $|x| = -2$.
    - **d.** $|\pi^2 - 9| < 0$.
    - **e.** The point $(1, 1)$ is inside the circle of radius 1 centered at the origin.
    - **f.** $\sqrt{x^4} = x^2$ for all real numbers x.
    - **g.** $\sqrt{a^2} < \sqrt{b^2}$ implies $a < b$ for all real numbers a and b.

46–48. Intervals to sets *Express the following intervals in set notation. Use absolute value notation when possible.*

46. $(-\infty, 12)$

47. $(-\infty, -2]$ or $[4, \infty)$

48. $(2, 3]$ or $[4, 5)$

49–50. Sets in the plane *Graph each set in the xy-plane.*

49. $\{(x, y): |x - y| = 0\}$

50. $\{(x, y): |x| = |y|\}$