Substitution

Ex: $\int \boxed{(2x+1)^3}\, dx$

Method 1: $(2x+1)^3 = 8x^3 + 3(2x)^2 \cdot 1 + 3 \cdot 2x \cdot 1^2 + 1^3$

$((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3)$ $\quad ||$

$$8x^3 + 12x^2 + 6x + 1$$

$$\int (8x^3 + 12x^2 + 6x + 1)\, dx$$

$$||$$

$$8 \cdot \frac{1}{3+1} x^{3+1} + 12 \cdot \frac{1}{1+2} x^{2+1} + 6 \frac{1}{1+1} x^{1+1} + x + C$$

$$= \underline{2x^4 + 4 \cdot x^3 + 3x^2 + x + C}$$

Method 2: $\underline{u = 2x+1} \Rightarrow du = \frac{du}{dx} dx = 2\, dx$

$$\int (2x+1)^3 dx = \int u^3 \frac{du}{2} = \frac{1}{2} \int u^3 du$$

$$= \frac{1}{2} \cdot \frac{1}{3+1} u^{3+1} + C$$

$$= \frac{1}{8} u^4 + C = \underline{\frac{1}{8}(2x+1)^4 + C}$$

$$\frac{1}{8}\left((2x)^4 + 4(2x)^3 + 6(2x)^2 + 4(2x) + 1\right) + C$$

$$||$$

$$\underline{2x^4 + 4 \cdot x^3 + 3x^2 + x + \boxed{\frac{1}{8} + C}}$$

Ex: $\int \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx$

$\boxed{u = \sqrt{x} = x^{\frac{1}{2}}}$ $du = \frac{du}{dx} \cdot dx = \frac{1}{2} x^{-\frac{1}{2}} dx = \frac{1}{2\sqrt{x}} dx$

$\Downarrow$

$\boxed{dx = 2\sqrt{x}\, du = 2u\, du}$

$= \int \frac{1}{u} e^{u} (2u) du$

$= \int 2e^{u} du = 2 \cdot e^{u} + C$

$= 2 \cdot e^{\sqrt{x}} + C$

$u = \sqrt{x} \Rightarrow \boxed{x = u^2}$ $\boxed{dx = \frac{dx}{du} du = 2u\, du}$

$\boxed{\int_1^4 \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx} = 2 \cdot e^{\sqrt{x}} \Big|_1^4 = 2 \cdot e^2 - 2e^1$

$\int_a^b f(x)\, dx = F(x) \Big|_a^b = F(b) - F(a)$

$u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du$

$\int_{\sqrt{1}}^{\sqrt{4}} \frac{1}{u} e^{u}\, 2u\, du = \int_1^2 2e^{u} du = 2 \cdot e^{u} \Big|_1^2$

$= 2e^2 - 2e$

Ex: $\int_0^2 \frac{3x^2}{(x^3+2)^3}\,dx$

$\boxed{u = x^3+2}$, $du = \frac{du}{dx}\cdot dx = \boxed{3x^2\,dx}$

2^3+2

$= \int_{2}^{10} \frac{du}{u^3} = \int_2^{10} u^{-3}\,du = \frac{1}{-3+1}\cdot u^{-3+1}\Big|_2^{10}$

0^3+2

$= -\frac{1}{2}u^{-2}\Big|_2^{10}$

$\frac{3}{25} = \frac{24}{200} = -\frac{1}{200} + \frac{1}{8}$ $= -\frac{1}{2}\cdot 10^{-2} - \left(-\frac{1}{2}\cdot 2^{-2}\right)$

$\frac{25}{200}$

$\int \frac{3x^2}{(x^3+2)^3}\,dx = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C$

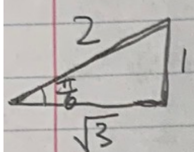
$= -\frac{1}{2}(x^3+2)^{-2} + C$

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\frac{1}{\cos x} = -\frac{-\sin x}{\cos^2 x} = \tan x \cdot \sec x.$$

Ex:

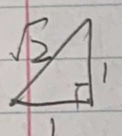
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan(x)\,\underbrace{\sec^2(x)\,dx}_{du} \qquad dx = \frac{du}{\sec^2 x}$$

$$u = \tan x, \quad du = \frac{du}{dx}\,dx = \sec^2 x\,dx$$



$$\int_{\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}}^{\tan(\frac{\pi}{4}) = 1} u \cdot du = \frac{1}{2}u^2 \Big|_{\frac{1}{\sqrt{3}}}^{1} = \frac{1}{2}1^2 - \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \boxed{\frac{1}{3}}$$



$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x \cdot dx}{\cos^3 x}$$

$$= \int \frac{\sin x}{(\cos x)^3}\,dx$$

$$v = \cos x, \quad dv = -\sin x \cdot dx$$

$$\int_{\cos\frac{\pi}{6}}^{\cos\frac{\pi}{4}} \frac{1}{v^3}(-dv) = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} v^{-3}\,dv$$

$$= -\frac{1}{-3+1}v^{-3+1}\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} = \frac{1}{2}\cdot \overbrace{v^{-2}}^{\frac{1}{v^2}}\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\left(\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\right) = \frac{1}{2}\cdot\left(2 - \frac{4}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$