Integrals

Indefinite Integral: $\int f(x)\,dx$ = Antiderivatives of f

$= F + C$ $\quad F' = f$

constant

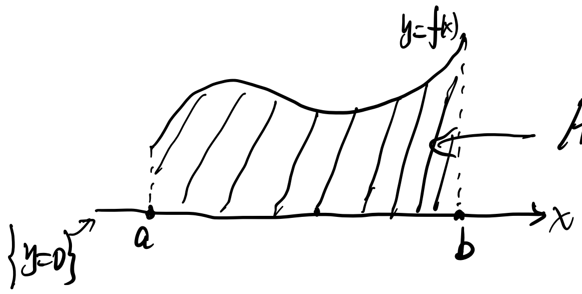
Ex: $\cdot \int x^p\,dx = \frac{1}{p+1}x^{p+1} + C, \quad p \neq -1$

$\cdot \int x^{-1}\,dx = \ln|x| + C.$

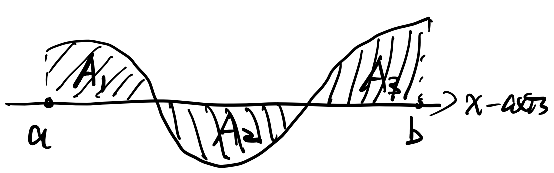
Definite Integral: $\int_a^b f(x)\,dx$ = net area

definite integral of $f(x)$ over $[a, b]$

integrand $\qquad$ limits of integration



Area $= \int_a^b f(x)\,dx$.

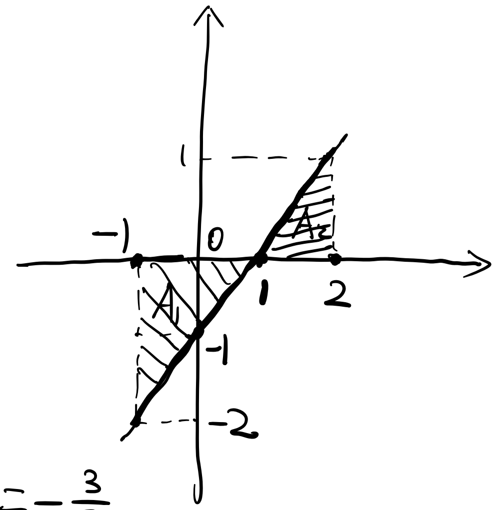


$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

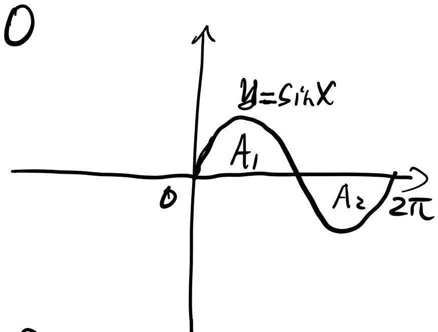
net area

$\int_a^b f(x)\,dx$ = net area of the region bounded between the x-axis and the graph of f over the interval $[a, b]$

Ex:
$$\int_{-1}^{2} \underbrace{(x-1)}_{f(x)=y}\, dx$$
$$= -A_1 + A_2$$
$$= -\tfrac{1}{2}\cdot 2\cdot 2 + \tfrac{1}{2}\cdot 1\cdot 1 = -2+\tfrac{1}{2} = -\tfrac{3}{2}$$

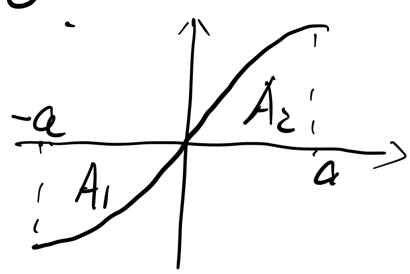


Ex:
$$\int_{0}^{2\pi} \sin(x)\,dx = A_1 - A_2 = 0$$



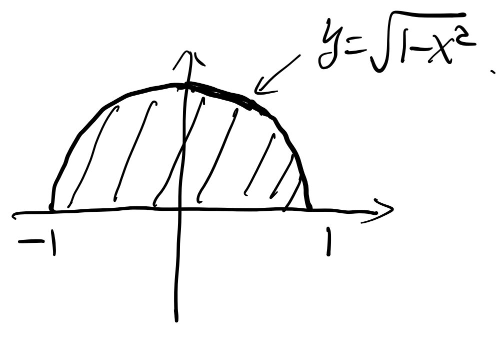
Ex: $\int_{-a}^{a} f(x)\,dx = -A_1 + A_2 = 0.$

$f(-x) = -f(x)$ odd function



Ex: $x^2+y^2=1$

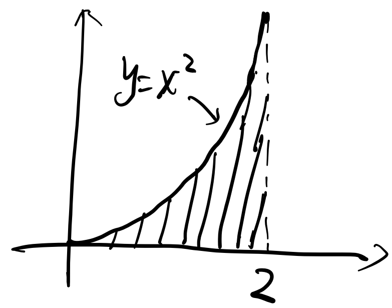
$\Rightarrow y = \sqrt{1-x^2}$

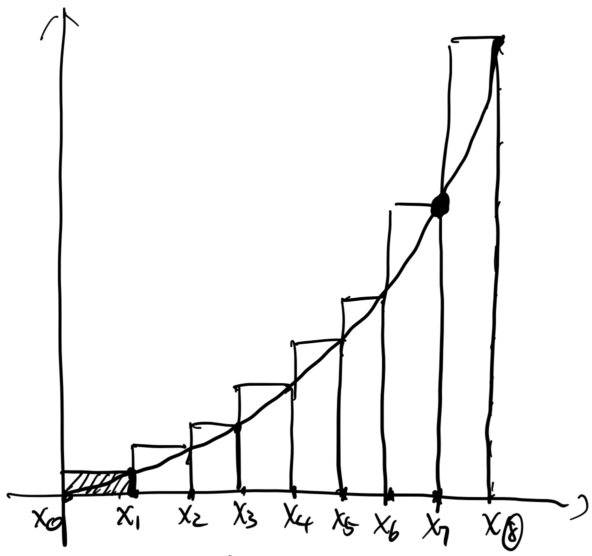


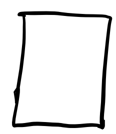
$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\pi\cdot 1^2 = \frac{\pi}{2}.$$

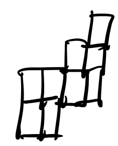
Ex: $\boxed{\int_0^2 x^2\,dx}$

$[0,2]$

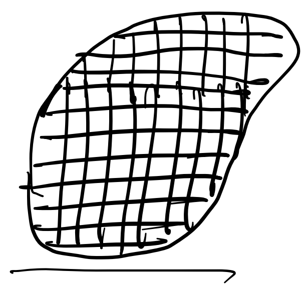








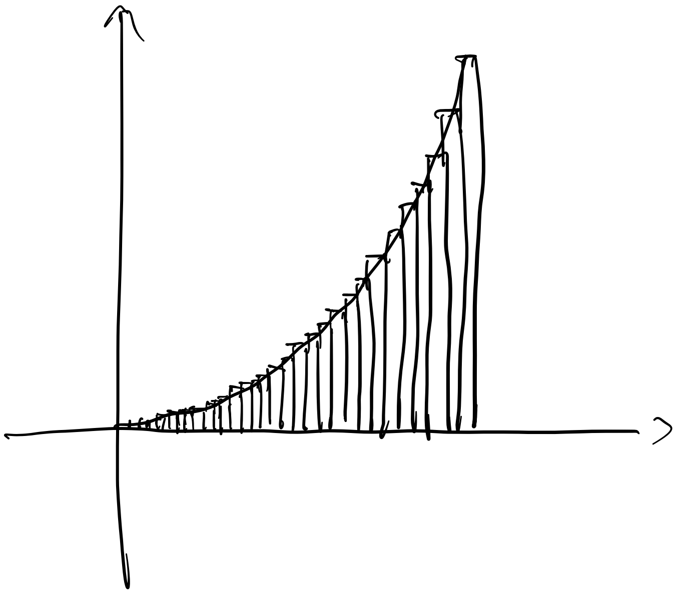
right Riemann sum



$$\Delta x = x_1 - x_0 = x_2 - x_1 = \cdots = x_8 - x_7$$

$$f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_8)\cdot\Delta x = \sum_{i=1}^{8} f(x_i)\cdot\Delta x$$

overestimate in this examples.



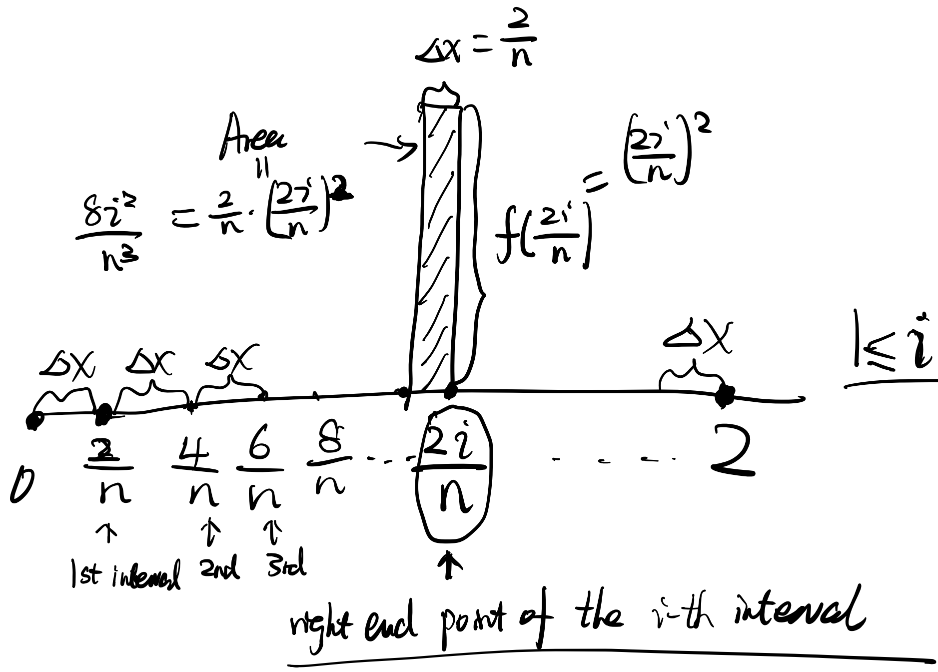
$$\int_1^2 \sin(x^3+x^2)\,dx$$

Divide $[0, 2]$ into n subintervals of equal size

(regular partition)

$$\boxed{\Delta x = \frac{2}{n}}$$

$f(x)=x^2$



$1 \leq i \leq n$

$$\left(\frac{2i}{n}\right)^2\cdot\frac{2}{n} = \frac{4i^2}{n^2}\cdot\frac{2}{n} = \frac{8\cdot i^2}{n^3}$$ area of the i-th rectangle

$$\sum_{i=1}^{n}\frac{8\cdot i^2}{n^3} = \frac{8\cdot 1^2}{n^3}+\frac{8\cdot 2^2}{n^3}+\cdots+\frac{8\cdot n^2}{n^3}$$

sum of areas of those n approximating rectangles.

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{8 \cdot i^2}{n^3} = \int_0^2 x^2 \, dx.$$

$$\parallel$$

$$\frac{8}{n^3} \sum_{i=1}^{n} i^2 = \frac{8}{n^3} \cdot (1^2+2^2+3^2+\cdots+n^2)$$

$$\parallel$$

$$\frac{1}{6} \cdot n \cdot (n+1)(2n+1)$$

$n=1$: $1^2 = 1 = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1+1)$

$n=2$: $1^2+2^2 = 5 = \frac{1}{6} \, 2 \cdot 3 \cdot 5 = 5$

$n=3$: $1^2+2^2+3^2 = 14 = \frac{1}{6} \, 3 \cdot 4 \cdot 7 = 14$

$$\lim_{n\to+\infty} \frac{8}{n^3} \cdot \frac{1}{6} \cdot n(n+1)(2n+1) = \frac{8}{6} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}$$

$$\parallel$$

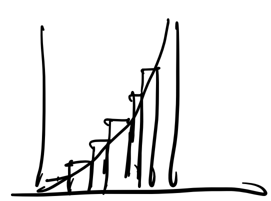
$$\lim_{n\to+\infty} \frac{8}{6} \cdot 1 \cdot \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

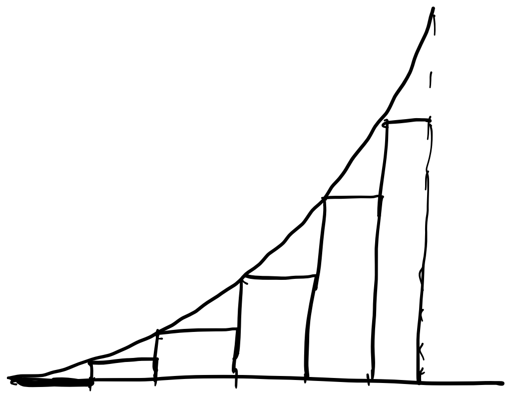
$$\parallel$$

$$\frac{8}{6} \cdot 2 = \frac{4}{3} \cdot 2 = \frac{8}{3}$$

$\parallel$ sum of ~~n-~~ area of n-rectangles

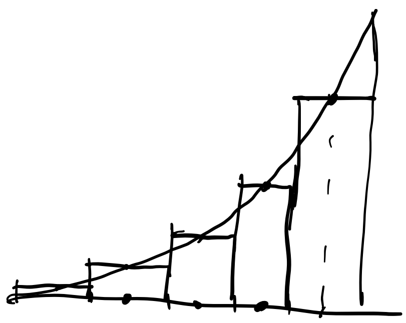
that approximate

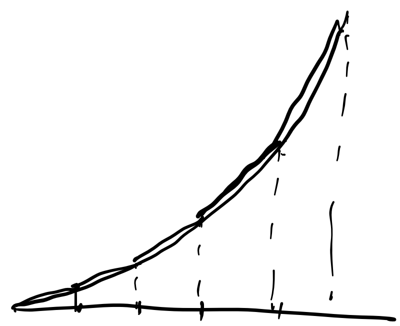




underestimate

left Riemann Sum





$$\int_0^2 x^2\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n}}_{f\left(\frac{2i}{n}\right)\cdot\Delta x} = \frac{8}{3}$$

$$\int x^2\,dx = \frac{1}{2+1}\cdot x^{2+1} + C = \frac{x^3}{3} + C.$$

Thm. $\int_a^b f(x)\,dx = F(b) - F(a)$. F is an antiderivative of f.

$$= \left(\int f(x)\,dx\right)\Big|_a^b$$

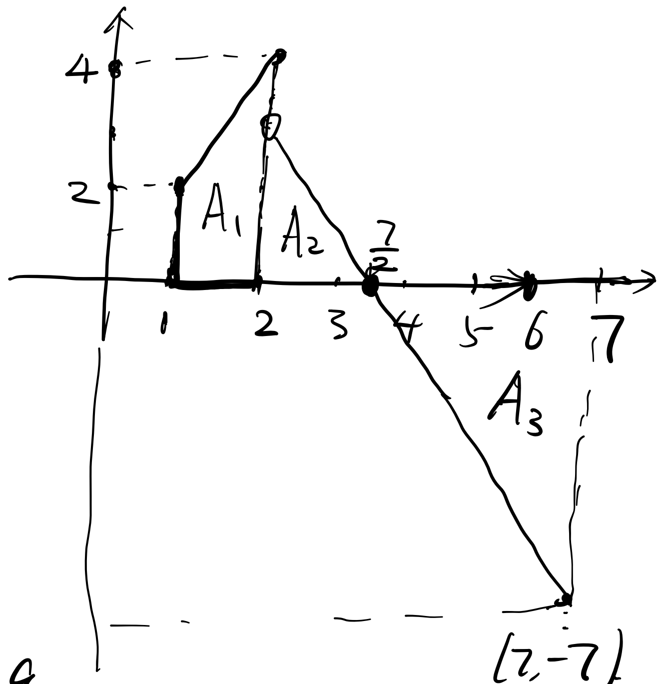
$$\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

Ex: $f(x) = \begin{cases} 2x & 1 \le x \le 2 \\ 7-2x & 2 < x \le 7 \end{cases}$

$$\int_1^7 f(x)\,dx$$

$$\shortparallel$$

$A_1 + A_2 - A_3$ = net area



A_1: $\frac{1}{2}\cdot(2+4)\cdot 1 = 3$

A_2: $\frac{1}{2}\cdot 3\cdot\left(\frac{7}{2}-2\right) = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4}$

A_3: $\frac{1}{2}\cdot 7\cdot\left(7-\frac{7}{2}\right) = \frac{7}{2}\cdot\frac{7}{2} = \frac{49}{4}$

$$\int_1^7 f(x)\,dx = 3 + \frac{9}{4} - \frac{49}{4} = 3 - \frac{40}{4} = 3 - 10 = -7$$

$$\int_1^7 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^7 f(x)\,dx$$

$$= \int_1^2 (2x)\,dx + \int_2^7 (7-2x)\,dx$$

$$= (x^2)\Big|_1^2 + (7x - x^2)\Big|_2^7$$

$$= (4-1) + \underbrace{(7\cdot 7 - 7^2)}_{=0} - \underbrace{(7\cdot 2 - 2^2)}_{=10}$$

$$= 3 - 10 = -7.$$