Antiderivatives

$(F+C)' = f$

F is an antiderivative of f if $F' = f$.

$f \longrightarrow$ antiderivative : $\int f(x)dx$ indefinite integral

$\uparrow$

a family of anti-derivatives

Ex:

$$\boxed{\int x^p \, dx = \frac{1}{p+1} x^{p+1} + C \qquad p \neq -1}$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$\ominus \frac{d}{dx}\cos(x) = +\sin(x)$

$$\int \cos(x)\, dx = \sin(x) + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$\Leftarrow$ $\frac{d}{dx}(\tan x) = \frac{d}{dx}\frac{\sin x}{\cos x}$

$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

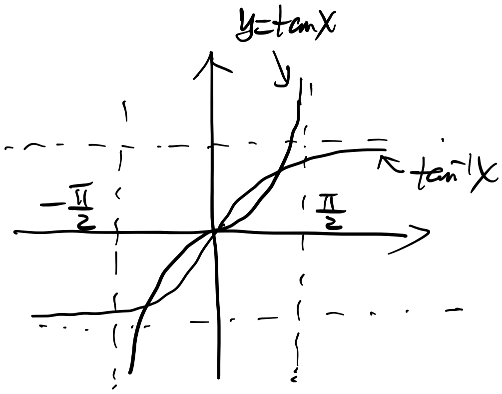
$\tan^{-1}x \neq \frac{1}{\tan x}$

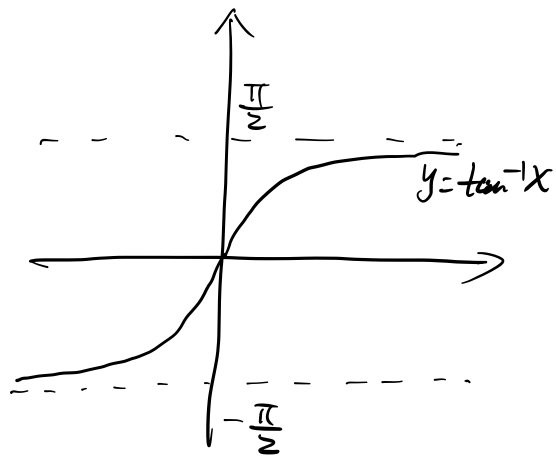
$\tan^{-1} x$ = inverse of $\tan x$.

$\frac{d}{dx}\frac{1}{\tan x} = \frac{d}{dx}\frac{\cos x}{\sin x} = -\csc^2 x$

$\Downarrow$

$$\boxed{\int \csc^2 x = -\frac{1}{\tan x} + C}$$





$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}}$$

$$\lim_{x \to +\infty} \tan^{-1} x = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1} x = \tan^{-1}(-\infty) = -\frac{\pi}{2}.$$

- $$\boxed{\begin{aligned} \int (f(x) + g(x))\,dx &= \int f(x)\,dx + \int g(x)\,dx \\ \int c \cdot f(x)\,dx &= c \cdot \int f(x)\,dx \end{aligned}}$$

$\int f(x)\,dx$ = indefinite integral = antiderivatives of f

$\int$ ↑ integral sign

$f(x)$ ↑ integrand

Ex: $\int \underline{\left(x+\frac{1}{x}\right)^2} dx = \int \left(x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2}\right) dx$

$= \int (x^2 + 2 + x^{-2})\, dx$

$= \int x^2 dx + \int 2\, dx + \int x^{-2} dx$

$$\underline{= \frac{1}{2+1} x^3 + 2x + \frac{1}{-2+1} x^{-2+1} + C}$$

$$\int \left(x+\frac{1}{x}\right)^2 dx$$

$a \frac{1}{0+1} x^{0+1} + C$

$\shortparallel$

$\int a \cdot x^0 dx$

$\shortparallel$

$\int a\, dx = ax + C$

$\uparrow$

constant

Ex: $\int \frac{2x^2}{x^2+1} dx$

$\shortparallel$

$\int \left(2 - \frac{2}{x^2+1}\right) dx$

$\shortparallel$

$\int 2\, dx - 2 \cdot \int \frac{1}{x^2+1} dx$

$\shortparallel$

$2x - 2 \cdot \tan^{-1} x + C$

$\frac{2x^2}{x^2+1} = \frac{2x^2+2-2}{x^2+1}$

$\shortparallel$

$\frac{2(x^2+1)-2}{x^2+1}$

$\shortparallel$

$$\boxed{2 - \frac{2}{x^2+1}}$$

Initial Value Problem (IVP):

$f(x) \rightsquigarrow$ Find an antiderivative F satisfying
$F(a) = b$.

Ex: Solve IVP:

$$g'(x) = \frac{1}{x}(1-x^3), \quad \boxed{g(1) = 10}.$$

$$\Rightarrow g(x) = \int \frac{1}{x}(1-x^3)\,dx$$
$$= \int \left(\frac{1}{x} - x^2\right) dx$$
$$= \int \frac{1}{x}\,dx - \int x^2\,dx = \ln|x| - \frac{1}{2+1}x^{2+1} + C$$
$$\boxed{g(x) = \ln|x| - \frac{1}{3}x^3 + C}.$$

$$\Rightarrow g(1) = \ln 1 - \frac{1}{3}1^3 + C = 10$$
$$= 0 - \frac{1}{3} + C = 10 \Rightarrow C = 10 + \frac{1}{3} = \frac{31}{3}$$

$$\Rightarrow g(x) = \ln|x| - \frac{1}{3}x^3 + \frac{31}{3}.$$

$$L'(x) = 3900 \cdot x^{-\frac{1}{2}}$$

$$(L'(4) = 1950, \quad L'(100) = 390)$$

$$\boxed{L(4) = 15600}$$

$$\Rightarrow L(x) = \int L'(x)\,dx = \int 3900 \cdot x^{-\frac{1}{2}}\,dx$$

$$= 3900 \cdot \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} = 3900 \cdot 2 \cdot x^{\frac{1}{2}} + C$$

$$= 7800 \cdot x^{\frac{1}{2}} + C$$

$$L(4) = 7800 \cdot 4^{\frac{1}{2}} + C = 7800 \cdot 2 + C$$

$$= 15600 + C$$

$$L(4) = 15600$$

$$\Rightarrow C = 15600 - 15600 = 0.$$

$$\Rightarrow L(x) = 7800 \cdot x^{\frac{1}{2}}$$

$$L(100) = 7800 \cdot 100^{\frac{1}{2}} = 7800 \cdot 10 = 78000.$$

Concave up $\Leftrightarrow f''>0 \Leftrightarrow f'$ is increasing

down $\Leftrightarrow f''<0 \Leftrightarrow f'$ is decreasing.

IVP: $C' = 3x^2 - 18x + 37$, $C(0) = 30$

$$C(x) = \int (3x^2 - 18x + 37)\,dx = \frac{3}{3}x^3 - \frac{18}{2}x^2 + 37x + C$$

$$= x^3 - 9x^2 + 37x + C$$

$$C(0) = \underline{C = 30}$$

$$\Rightarrow C(x) = x^3 - 9x^2 + 37x + 30$$

$$\Rightarrow C(3) = 3^3 - 9\cdot 3^2 + 37\cdot 3 + 30 = 27 - 81 + 111 + 30$$

$$= 27 + 60 = 87$$