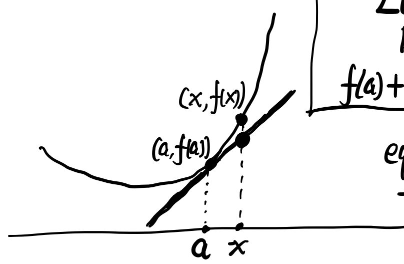
Linear Approximation



$$L(x) = f(a) + f'(a)\cdot(x-a)$$

equation for the tangent line at point $(a, f(a))$

$y = f(x)$

Linear Approximation to $y = f(x)$ at point a: $\underline{L(x) = f(a) + f'(a)\cdot(x-a)}$

Error $= f(x) - L(x)$

Ex: $\sqrt{82} = \sqrt{x}\big|_{x=82}$ $\quad f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f(82)$.

$f(81) = \sqrt{81} = 9$,

$L(x) = f(81) + f'(81)\cdot(x-81)$
$\quad = f(a) + f'(a)\cdot(x-a)$ $\quad \underline{a=81}$ near 82

$f'(x) = \frac{1}{2}\cdot x^{-\frac{1}{2}}$. $\quad f'(81) = \frac{1}{2}\cdot(81)^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{81}} = \frac{1}{18}$

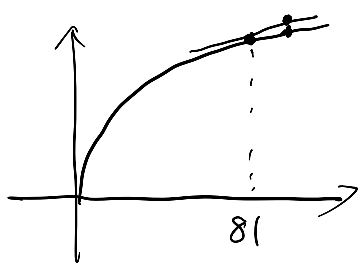
$L(x) = 9 + \frac{1}{18}\cdot(x-81)$

$L(82) = 9 + \frac{1}{18}\cdot(82-81) = 9 + \frac{1}{18} = 9 + 0.056 = 9.05556$

$\sqrt{82} = 9.05538$ $\qquad$ Error $= \frac{\text{approximate} - \text{exact}}{\approx + 0.00018}$

overestimate

concave down



Ex: $\ln(0.93)$ $\quad f(x)=\ln x$, $f(0.93)$

$a=1$, $f(1)=\ln(1)=0$.

$f'(x)=\frac{1}{x}$, $f'(a)=f'(1)=1$.

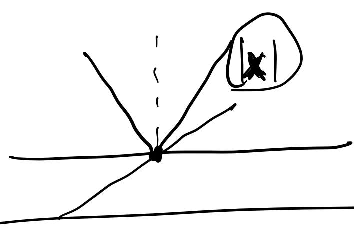
$$L(x)=f(a)+f'(a)(x-a)$$
$$=f(1)+f'(1)(x-1)=0+1\cdot(x-1)=x-1$$

$$L(0.93)=0.93-1=\underline{-0.07}$$

$\ln(0.93)=-0.07257\ldots$ $\quad$ Error = Approx. − Exact $=0.00257\ldots$

$\ln(0.93)$

- $L(x)=f(a)+f'(a)\cdot(x-a)$ works only for differentiable functions



$$\Delta L=L(x)-f(a)=f'(a)\cdot\underbrace{(x-a)}_{=\Delta x}$$

$f(x)-f(a)=\Delta y$

$$P(z) = 1000 \cdot e^{-\frac{z}{10}}.$$

$z: 9 \to 9.03$ km

$\Delta z = 9.03 - 9 = 0.03.$

$$\Delta L = P'(a) \cdot \Delta z$$

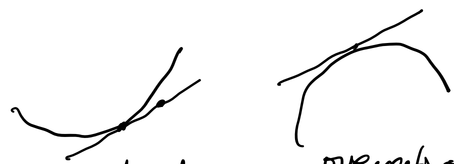
$\Delta p \sim \Delta L$

$$P'(z) = 1000 \cdot e^{-\frac{z}{10}} \cdot \left(-\frac{1}{10}\right) = -100 \cdot e^{-\frac{z}{10}}$$

$$P'(a) = -100 \cdot e^{-\frac{9}{10}}$$

$$\Delta L = -100 \cdot e^{-\frac{9}{10}} \cdot (0.03) = -3 e^{-\frac{9}{10}}$$

$$P''(z) = -100 \cdot e^{-\frac{z}{10}} \cdot \left(-\frac{1}{10}\right) = 10 e^{-\frac{z}{10}} > 0$$



(A) $C(41) = 2100 + 90 \times 41 - 0.2 \cdot 41^2$

$= 2100 + 3690 - 336.2 = 5453.8$

$C(40) = 2100 + 90 \cdot 40 - 0.2(40)^2$

$= 2100 + 3600 - 0.2 \cdot 40^2.$

$a^2 - b^2 = (a+b)(a-b)$

$\Delta C = C(41) - C(40) = 90 - 0.2((41)^2 - (40)^2)$

$= 90 - 0.2 \cdot 81 \cdot 1$

$= 90 - 16.2 = 73.8$

(B) $C'(x) = 0 + 90 - 0.2 \cdot 2 \cdot x = 90 - 0.4x.$

$$\Delta \underset{\substack{\uparrow \\ \text{Linear app.}}}{L} = C'(a) \cdot \Delta x = C'(40) \cdot (41 - 40)$$

$$= (90 - 0.4 \cdot 40) \cdot 1$$

$$= 90 - 16 = \underline{74}.$$

$$\Updownarrow$$

$$L(x) = L(a) + C'(a) \cdot (x - a)$$

$$\underbrace{L(x) - L(a)}_{\Delta L} = C'(a) \cdot \underbrace{(x - a)}_{\Delta x}$$

$\underbrace{y = f(x)}_{\wr}$, pick a in the domain

f is differentiable near a

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

$$dy = \boxed{\underbrace{\Delta L}_{\substack{\wr \\ \Delta y = f(x) - f(a)}} = \underbrace{f'(a)}_{\wr} \cdot \Delta x} = f' dx$$

differential $\| $ $dy = f' dx$

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differential of f: $\boxed{df = f'(x) \cdot dx}.$

$$\frac{df}{\cancel{dx}} \cancel{dx}$$

- Antiderivative $\quad f \xrightarrow{\text{derivative}} f' = g$

Def: A function G is an antiderivative of $g(x)$ if $G' = g$

$g(x) = x^3 \Rightarrow$ antiderivative $G = \frac{1}{4}x^4$

$G' = \frac{1}{4} \cdot 4 \cdot x^3 = x^3$

$\frac{1}{4}x^4 + C$

$\boxed{g(x) = 0} \Rightarrow$ antiderivative $\boxed{G(x) = C}$

$g(x) = x^p \Rightarrow$ antiderivative $G(x) = \frac{1}{p+1} x^{p+1} + C$

$G' = \frac{1}{\cancel{p+1}} \cancel{p+1} \cdot x^p$

$f(x) = \sin x \Longrightarrow$ antiderivative : $F(x) = -\cos x + C$

$F'(x) = +\sin x$

$$f(t) = \frac{1}{t} \quad t \neq 0 \quad \xRightarrow{\text{antiderivative}} \quad \boxed{\ln t + C, \ t > 0}$$

$$\frac{1}{t} = t^{-1}$$

$$\cancel{\frac{1}{-1+1} t^{-1+1} = \frac{1}{0} t^{0}}$$

$$\boxed{t^{p} \Longrightarrow \frac{1}{p+1} t^{p+1} + C \qquad (p \neq -1)}$$

$$\frac{1}{t} \xRightarrow{\text{antiderivative}} \ln(-t) + C, \quad t < 0$$

$$\frac{d}{dt} \ln(-t) = \frac{1}{-t} \cdot (-1) = \frac{1}{t}$$

$$\frac{1}{t} \Longrightarrow \text{antiderivative} \quad \ln|t| + C$$