L'Hôpital Rule.

f and g are differentiable

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, then ← indeterminate form

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ provided the limit on the right exists.

$$\left(\frac{f}{g}\right)' \neq \lim_{x\to a} \frac{f(x)}{g(x)} \neq \lim_{x\to a} \left(\frac{f}{g}\right)'$$

- $\lim_{x\to -1} \frac{5x^4+7x^3+9x+11}{x+1} = \frac{5-7-9+11=0}{0}$

$$\lim_{x\to -1} \frac{20\cdot x^3+21x^2+9+0}{1+0} = 20\cdot(-1)^3+21\cdot(-1)^2+9$$
$$= -20+21+9 = \underline{10}$$

- $\lim_{x\to 0} \frac{1-\cos(2x)}{4x^2} = \frac{1-\cos(0)}{4\cdot 0^2} = \frac{0}{0}$ ← indeterminant form

$$\lim_{x\to 0} \frac{0+\sin(2x)\cdot 2}{8x} = \lim_{x\to 0} \frac{2\cdot \sin(2x)}{8x} = \frac{0+0}{8\cdot 0} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{2\cdot\cos(2x)\cdot 2}{8} = \frac{4\cdot\cos(0)}{8} = \frac{4}{8} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{1-\sin x}{2-\sin x} = \frac{1}{2} \neq \frac{0}{0}$$

$$\neq$$

$$\lim_{x\to 0} \frac{0-\cos x}{0-\cos x} = \frac{0-1}{0-1} = 1.$$

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indeterminate form $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\ln(2x+6)}{\ln(4x+5)+9} = \frac{\infty}{\infty}$$

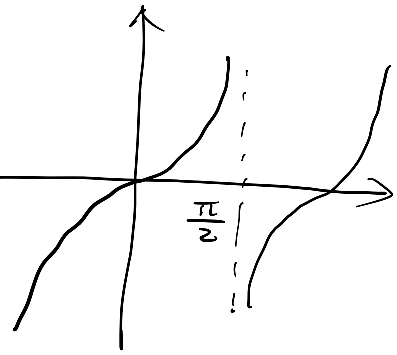
$$\lim_{x\to\infty} \frac{\frac{1}{2x+6}\cdot 2}{\frac{1}{4x+5}\cdot \frac{4}{2}+0} = \lim_{x\to\infty}\left(\frac{4x+5}{2x+6}\cdot\frac{1}{2}\right)$$

$$= \lim_{x\to\infty} \frac{\cancel{x}(4+\frac{5}{x})}{\cancel{x}(2+\frac{6}{x})}\cdot\frac{1}{2}$$

$$1 = \frac{1}{2}\,\frac{4+0}{2+0} = \lim_{x\to\infty} \frac{4+\frac{5}{x}}{2+\frac{6}{x}}\cdot\frac{1}{2}$$

Ex: $\lim\limits_{x\to(\frac{\pi}{2})^-} \boxed{\dfrac{\tan x}{3/(2x-\pi)}} = \dfrac{\tan(\frac{\pi}{2})}{-\frac{3}{0}} = \dfrac{+\infty}{-\infty}$

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\sec^2 x = \frac{1}{\cos^2 x}}{-\frac{6}{(2x-\pi)^2}} = -\frac{1}{6}\frac{(2x-\pi)^2}{\cos^2 x}$$



$$\left(\frac{3}{2x-\pi}\right)' = \frac{0\cdot(2x-\pi)-3\cdot(2)}{(2x-\pi)^2} = -\frac{6}{(2x-\pi)^2}$$

$$\lim_{x\to(\frac{\pi}{2})^-} -\frac{1}{6}\frac{(2x-\pi)^2}{\underbrace{\cos^2 x}_{(\cos x)^2}} = -\frac{1}{6}\cdot\frac{0^2}{0^2}$$ ← indeterminate form.

$$\ominus\frac{1}{6}\cdot\lim_{x\to(\frac{\pi}{2})^-}\frac{2\cdot(2x-\pi)\cdot\cancel{2}}{\cancel{2}\cdot\cos x\cdot(-\sin x)} = \frac{2}{6}\lim_{x\to(\frac{\pi}{2})^-}\frac{2x-\pi}{\cos x\cdot\sin x} = \frac{1}{3}\cdot\frac{0}{0}$$

indeterminant

$$\frac{1}{3}\lim_{x\to(\frac{\pi}{2})^-}\frac{2x-\pi}{\cos x\cdot\sin x} = \frac{1}{3}\lim_{x\to(\frac{\pi}{2})^-}\frac{2}{-\sin x\cdot\sin x+\cos x\cdot\cos x}$$

$$= \frac{1}{3}\cdot\frac{2}{-1\cdot1+0\cdot0} = -\frac{2}{3}$$

$$\lim_{x\to(\frac{\pi}{2})^-} -\frac{1}{6}\frac{(2x-\pi)^2}{\cos^2 x}$$

$$x-\frac{\pi}{2}=t$$

$$2x-\pi=2t$$

$$=\lim_{t\to 0} -\frac{1}{6}\cdot\frac{(2t)^2}{\cos^2(\frac{\pi}{2}+t)}$$

$$\boxed{\cos\left(\frac{\pi}{2}+t\right)=-\sin t}$$

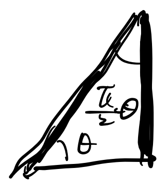
$$=\lim_{t\to 0} -\frac{1}{6}\cdot\frac{4t^2}{\sin^2 t}=-\lim_{t\to 0}\frac{2}{3}\cdot\frac{t^2}{\sin^2 t}$$

$$=-\frac{2}{3}\cdot 1=-\frac{2}{3}$$

$$\lim_{t\to 0}\frac{\sin t}{t}=1$$

$$\lim_{t\to 0}\left(\frac{t}{\sin t}\right)^2=1^2=1$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$



$$\cos\left(\frac{\pi}{2}+\theta\right)=\sin(-\theta)=-\sin\theta$$

$$\frac{\tan x}{3/(2x-\pi)} = \frac{\frac{\sin x}{\cos x}}{\frac{3}{2x-\pi}} = \frac{(2x-\pi)\cdot \sin x}{3\cdot \cos x}$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\tan x}{3/(2x-\pi)} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{(2x-\pi)\sin x}{3\cdot \cos x} = \frac{0\cdot 1}{3\cdot 0} = \frac{0}{0}$$

indeterminate form

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{2\cdot \sin x + (2x-\pi)\cos x}{-3\cdot \sin x} = \frac{2\cdot 1+0}{-3\cdot 1} = -\frac{2}{3}$$

Ex: $\lim_{x\to 1^+} x^{\frac{3}{1-x}} = 1^{\infty}$ ← indeterminate form

$$\lim_{x\to 1^+} \ln\left(x^{\frac{3}{1-x}}\right) = \lim_{x\to 1^+} \frac{3}{1-x}\cdot \ln x = \frac{3\cdot \ln x}{1-x}$$

$$\| \qquad = \frac{3\cdot \ln 1}{1-1} = \frac{0}{0}$$

$$\lim_{x\to 1^+} \frac{3\cdot\frac{1}{x}}{0-1} = \frac{3}{-1} = -3$$

$$\lim_{x\to 1^+} x^{\frac{3}{1-x}} = e^{\lim_{x\to 1^+} \ln x^{\frac{3}{1-x}}} = e^{-3}$$

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} \ln f^{g}} \quad \lim_{x\to a} \ln f^{g} = \lim_{x\to a} g\cdot \ln f$$

$$\lim_{x\to a} \frac{f}{g}$$

$$\lim_{x\to\frac{\pi}{2}} (2x-\pi)^2\cdot\sec^2 x = \lim_{x\to\frac{\pi}{2}} \frac{(2x-\pi)^2}{\cos^2 x}$$

Ex:

$$\underbrace{\lim_{x\to 10^+}\left(\frac{1}{\ln(x-9)} - \frac{1}{x-10}\right)}_{=} = \underbrace{\frac{1}{0}-\frac{1}{0}}_{= \infty-\infty}$$

$$\lim_{x\to 10^+} \frac{x-10-\ln(x-9)}{\underbrace{(x-10)\cdot\ln(x-9)}} = \frac{0-0}{0\cdot 0} = \frac{0}{0}$$

$$\lim_{x\to 10^+} \frac{1-\frac{1}{x-9}}{\frac{1}{x-9}(x-10)+\ln(x-9)\cdot 1} = \lim_{x\to 10^+}\frac{x-10}{\underbrace{(x-10)+\ln(x-9)}_{=\frac{0}{0}}}$$

$$1\cdot\ln(x-9)+(x-10)\cdot\frac{1}{x-9}$$

$$\lim_{x\to 10^+} \frac{1}{1+\frac{1}{x-9}} = \frac{1}{1+\frac{1}{1}} = \frac{1}{2} = \frac{1}{2}.$$

$\underline{Ex}$:

$$\lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$\lim_{x\to+\infty} \frac{\cancel{\frac{1}{2}}(x+1)^{-\frac{1}{2}}}{\cancel{\frac{1}{2}}x^{-\frac{1}{2}}} = \frac{0}{0}$$

$$\lim_{x\to+\infty} \frac{\cancel{-\frac{1}{2}}(x+1)^{-\frac{3}{2}}}{\cancel{-\frac{1}{2}}x^{-\frac{3}{2}}} = \frac{0}{0}$$

$$\boxed{\lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x}}}{\sqrt{1}} = \frac{1}{1} = 1}$$

$x^p \quad p>0$

e^x

$\ln x$, $\ln^p x$, $x^p \ln^q x$

x^x

$$\lim_{x\to+\infty} \frac{x^p}{e^x} = \lim_{x\to+\infty} \frac{p\cdot x^{p-1}}{e^x} = \lim_{x\to+\infty} \frac{p\cdot(p-1)\cdot x^{p-2}}{e^x}$$

$$= \lim_{x\to+\infty} \frac{p(p-1)(p-2)x^{p-3}}{e^x} \cdots$$

$$= \cdots\cdots = \begin{cases} \dfrac{p\cdot(p-1)\cdots 0}{e^x} = 0 & p \text{ is an integer} \\ p(p-1)\cdots(p-[p])\cdot\dfrac{x^{p-[p]-1}}{e^x} & [p]\text{: longest integer smaller than } p. \end{cases}$$

$$\frac{x^{p-[p]-1}}{e^x} = \frac{1}{x^{[p]+1-p}\cdot e^x} \longrightarrow \frac{1}{\infty} = 0$$

$\Rightarrow e^x$ grows faster than x^p, $p>0$.