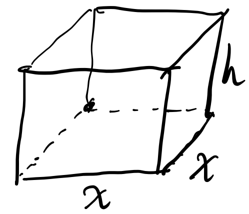
Question 1: (Baggage Volume)

1. Draw picture and identify the variables



2. objective function:

$$\boxed{V = x^2 \cdot h}$$

3. Constraint: $x+x+h \leq 72 \rightsquigarrow \boxed{2x+h=72}$

4. eliminate all but one independent variable:

$$2x+h=72 \Rightarrow h=72-2x$$

5. $$V(x) = x^2\cdot(72-2x) = 72\cdot x^2 - 2x^3$$

6. Find absolute maximum on the interval of interest:

$$x \geq 0, \quad h = 72-2x \geq 0 \Rightarrow x \leq 36. \quad x \in [0, 36].$$

$$V'(x) = \underset{\overset{\parallel}{6x\cdot(24-x)}}{144\cdot x - 6x^2} = 0 \Rightarrow x = 0,\ 24. \quad \text{critical points.}$$

$(2^3\cdot 3)^3 = 2^9\cdot 3^3 = 512\cdot 27$

$$V(0)=0, \quad V(24) = 24^2\cdot(72-\underbrace{2\cdot 24}_{48}) = 24^2\cdot 24 = 24^3$$

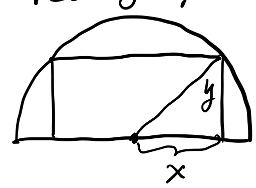
$$\begin{array}{r} 512 \\ \times\ 27 \\ \hline 3584 \\ 1024 \\ \hline 13824 \end{array}$$

End points: $V(0)=0$, $V(36) = 36^2\cdot 0 = 0$.

$\Rightarrow$ absolute maximum: $V(24) = 13824$

obtained at $x=24$, $h = 72-24\cdot 2 = 24$

Question 2: (Area of rectangle)

1. 

2. $A = 2x \cdot y$ Objective function

3. $x^2 + y^2 = 8^2 = 64$ constraint.

4. $y = \sqrt{64 - x^2}$ elimination

5. $A(x) = 2x \cdot \sqrt{64 - x^2}$

6. $x \in [0, 8]$

$$A'(x) = 2 \cdot \sqrt{64-x^2} + 2x \cdot \frac{1}{2}(64-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= 2\sqrt{64-x^2} - \frac{2x^2}{\sqrt{64-x^2}} = \frac{2}{\sqrt{64-x^2}}(64-x^2-x^2) = \frac{2}{\sqrt{64-x^2}}(64-2x^2) = 0$$

$\Rightarrow 2x^2 = 64 \Rightarrow x^2 = 32 \Rightarrow x = \sqrt{32} = 4\sqrt{2}$ critical point.

Value at critical point: $A(4\sqrt{2}) = 2 \cdot 4\sqrt{2} \cdot \sqrt{64-32} = 2 \cdot 4\sqrt{2} \cdot 4\sqrt{2} = 64.$

at End points: $A(0) = 0 = A(8)$

$\Rightarrow$ absolute maximum: $A(4\sqrt{2}) = 64.$

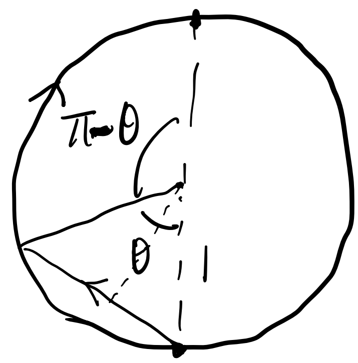
$\Rightarrow$ base $= 2x = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$

height $= y = \sqrt{64 - (4\sqrt{2})^2} = \sqrt{32} = 4\sqrt{2}.$

Question 3: Example 3 (Walking and swimming)

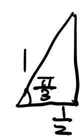
swimming speed: 2 mi/h

Walking speed: 4 mi/h



$$T(\theta) = \frac{2\cdot\sin\left(\frac{\theta}{2}\right)}{2} + \frac{\pi-\theta}{4}, \quad 0 \le \theta \le \pi$$

$$= \sin\left(\frac{\theta}{2}\right) + \frac{\pi-\theta}{4}$$



Find critical points:

$$T'(\theta) = \cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{2} - \frac{1}{4} = 0 \Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \frac{\theta}{2} = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow \theta = \frac{2}{3}\pi.$$

$$T''(\theta) = -\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{4}\sin\left(\frac{\theta}{2}\right) < 0 \text{ in } (0,\pi)$$

$\Rightarrow$ graph of T is concave

$$T''\left(\frac{2}{3}\pi\right) = -\frac{1}{4}\sin\left(\frac{\pi}{3}\right) < 0$$

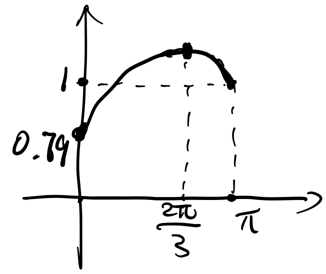
$\Rightarrow \theta = \frac{2}{3}\pi$ is a local maximum values at critical pt:

$$T\left(\frac{2}{3}\pi\right) = \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{12} = \frac{\sqrt{3}}{2} + \frac{\pi}{12}$$

$$\approx 0.87 + 0.26 = 1.13$$

at end points:

$$T(0) = \frac{\pi}{4} \approx 0.79, \quad T(\pi) = 1$$



(absolute)

$\Rightarrow$ minimum time $T(0) = \frac{\pi}{4}$ (by just walking)

maximum time $T\left(\frac{2}{3}\pi\right) \sim 1.13$ hour

- Study how the minimum time changes when the walking speed v changes:

If the walking speed is v, then the ~~total~~ time function:

$$T(\theta) = \sin\left(\frac{\theta}{2}\right) + \frac{\pi - \theta}{v}$$

$$= \text{swimming time} + \text{Walking time}$$

$$\Rightarrow T'(\theta) = \frac{1}{2}\cos\left(\frac{\theta}{2}\right) - \frac{1}{v} = 0 \Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{2}{v}$$

$$0 \le \theta \le \pi \Rightarrow 0 \le \frac{\theta}{2} \le \frac{\pi}{2} \Rightarrow 0 \le \cos\left(\frac{\theta}{2}\right) \le 1.$$

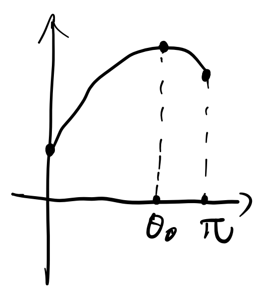
$\Rightarrow$ case 1: $v \geq 2$, there is a critical point at $\theta_0 = 2\cdot\cos^{-1}\left(\frac{2}{v}\right)$ which is a local maximum

$$T(0) = \frac{\pi}{v}, \quad T(\pi) = 1$$

all by walking

all by swimming

case 1(a): $v \geq \pi$, $\overset{\frac{\pi}{v}}{\overset{\parallel}{T(0)}} < \overset{1}{\overset{\parallel}{T(\pi)}}$



absolute minimum time by walking

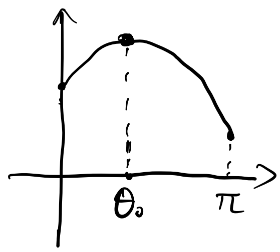
(because of walking fast)

(because of walking slower)

absolute minimum by swimming

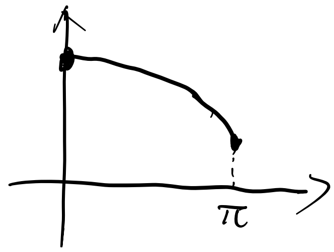
Case 1(b): $2 < v < \pi$, $T(0) > T(\pi)$

$$\frac{\pi}{v} > 1$$



Case 1(c) $v = 2$. $T(0) > T(\pi)$

absolute minimum by swimming



(walking really slow)

case 2: $0 < v < 2 \Rightarrow T'(\theta) = \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) - \frac{2}{v}\right) < 0$

$\Rightarrow$ $T(\theta)$ is decreasing and no critical point

$\Rightarrow$ absolute minimum: $T(\pi) = 1$

maximum: $T(0) = \frac{\pi}{v} > T(\pi) = 1$

swimming