Implicit Differentiation

$F(x, y) = 0 \rightsquigarrow y = y(x)$

Ex: $\sin(x \cdot y) + y = 0$ $\quad y = y(x)$?

$\frac{dy}{dx}$

$$\frac{d}{dx}\left(\sin(x \cdot y) + y\right) = \frac{d}{dx} 0 = 0$$

$$= \frac{d}{dx}\sin(x \cdot y(x)) + \frac{d}{dx} y$$

$$\frac{d}{dx}\sin(x \cdot y(x)) = \cos(x \cdot y) \cdot \frac{d}{dx}(x \cdot y(x)) = \cos(x \cdot y) \cdot [1 \cdot y + x \cdot y']$$

$$\cos(x \cdot y) \cdot (y + x \cdot y') + y' = 0$$

$$= \cos(xy) \cdot y + \cos(xy) \cdot x \cdot y' + y' = \cos(xy) \cdot y + y' \cdot (\cos(xy) \cdot x + 1) = 0$$

$$y' (\cos(xy) \cdot x + 1) = -\cos(xy) \cdot y$$

$$\Rightarrow y' = -\frac{\cos(xy) \cdot y}{\cos(xy) + 1}, \quad y' = \frac{dy}{dx}$$

$\frac{dy}{dx}$ is a function of both x and y.

Ex: $e^{x+y} = \sqrt{x+y}$ $\quad$ find $\frac{dy}{dx} = ?$

$$\frac{d}{dx} x^c = c \cdot x^{c-1}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^{7x} = e^{7x} \cdot \underbrace{7}_{\frac{d}{dx}(7x)}$$

$$\frac{d}{dx} e^{x+y} = e^{x+y} \frac{d}{dx}(x+y) = e^{x+y} \cdot (1 + y').$$

$$\frac{d}{dx}\sqrt{x+y} = \frac{1}{2}(x+y)^{-\frac{1}{2}}\cdot\underset{\frac{d}{dx}(x+y)}{\underset{\shortparallel}{(1+y')}}$$

$$e^{x+y}\cdot(1+y') = \frac{1}{2}(x+y)^{-\frac{1}{2}}\cdot(1+y')$$

$a(b+c)=ab+ac.$

$$e^{x+y}+e^{x+y}\cdot y' = \frac{1}{2}(x+y)^{-\frac{1}{2}} + \frac{1}{2}(x+y)^{-\frac{1}{2}}\cdot y'$$

$$\left[e^{x+y}-\frac{1}{2}(x+y)^{\frac{1}{2}}\right]\cdot y' = \frac{1}{2}(x+y)^{-\frac{1}{2}} - e^{x+y}$$

$$\Rightarrow y' = \frac{\frac{1}{2}(x+y)^{-\frac{1}{2}} - e^{x+y}}{e^{x+y}-\frac{1}{2}(x+y)^{-\frac{1}{2}}} = -1$$

Ex $\quad x^3 - \sqrt{y+1} = \ln(y) \qquad \frac{dy}{dx} = ?$

$$\frac{d}{dx}\left(x^3 - \sqrt{y+1}\right) = 3x^2 - \frac{1}{2}(y+1)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(y+1)$$

$\sqrt{y+1} = \frac{d}{dx}(y+1)^{\frac{1}{2}}$

$$= 3x^2 - \frac{1}{2}\cdot\frac{1}{\sqrt{y+1}}\,y'$$

$$\frac{d}{dx}\ln y = \frac{1}{y}\cdot\frac{d}{dx}y = \frac{1}{y}y'$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

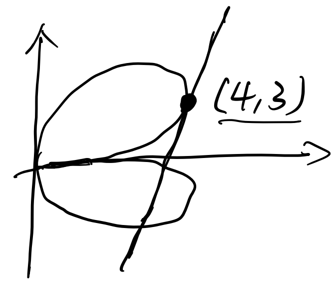
$$3x^2 - \frac{1}{2\sqrt{y+1}}y' = \frac{y'}{y} \Rightarrow 3x^2 = \frac{1}{2\sqrt{y+1}}y' + \frac{1}{y}\cdot y'$$

$$= \left(\frac{1}{2\sqrt{y+1}} + \frac{1}{y}\right)y'$$

$$\frac{y+2\sqrt{y+1}}{2y\cdot\sqrt{y+1}}$$

$$y' = 3x^2\cdot\frac{2y\cdot\sqrt{y+1}}{y+2\sqrt{y+1}} = (3x^2)\cdot\frac{2y\cdot\sqrt{y+1}}{y+2\sqrt{y+1}}$$

$$\frac{y}{y(2\sqrt{y+1})} + \frac{2\sqrt{y+1}}{y(2\sqrt{y+1})}$$



Ex: $\boxed{36(x^2+y^2)^2 = 625\cdot x\cdot y^2}$

$\boxed{36(4^2+3^2)^2 = 625\cdot 4\cdot 3^2} \Rightarrow \boxed{(4,3)}$ is on this curve.

$$\frac{dy}{dx} \qquad \underbrace{72\cdot(x^2+y^2)^1\cdot(2x+2y\cdot y')}_{=\ 36\cdot 2\cdot(x^2+y^2)^{2-1}\cdot\frac{d}{dx}(x^2+y^2)}$$

$$\frac{d}{dx}(y^2) = \underbrace{\left(\frac{d}{dy}y^2\right)\cdot\frac{d}{dx}y}_{=\ 2y\cdot\frac{dy}{dx}}$$

$$\frac{d}{dx}y = y'$$

$$\frac{d}{dx}(625\cdot x\cdot y^2) = 625\frac{d}{dx}(x\cdot y^2) = 625\cdot(y^2 + x\cdot 2y\cdot y')$$

$$1\cdot y^2 + x\cdot\frac{d}{dx}y^2$$

$$72(x^2+y^2)\cdot(2x+2y\cdot y') = 625(y^2+2xy\cdot y')$$

$$72\cdot(4^2+3^2)\cdot(2\cdot 4+2\cdot 3\cdot y') = 625(3^2+2\cdot 4\cdot 3\cdot y')$$

$$72\cdot\underbrace{5^2}_{=25}\cdot(8+6y') = \underbrace{625}_{=25^2}\cdot(9+24\cdot y')$$

$$72\cdot(8+6y') = 25(9+24\cdot y').$$

$$72\cdot 8 - 25\cdot 9 = (25\cdot 24 - 72\cdot 6)\,y'$$

$72 = 24\cdot 3$

$8^2\cdot 9$

$= 9\cdot(64-25) = 9\cdot 39$

$24\cdot(25-18) = 24\cdot 7\cdot y'$

$$y' = \frac{9\cdot 39}{24\cdot 7} = \frac{3\cdot 39}{8\cdot 7} = \frac{117}{56} \quad \underline{\text{slope}}$$

pt. $(4,3)$.

<u>tangent line</u>: $y - y_1 = k(x - x_1)$

$y = kx + b.$

$$\underline{y - 3 = \frac{117}{56}(x-4)}.$$

Ex

$\sqrt{x^2+4} \neq x+2$

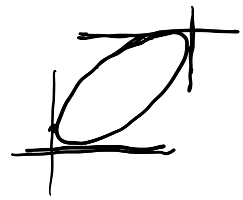
$$\lim_{x\to\infty}\frac{\sqrt{x^2+\cancel{4}}}{2x} = \lim_{x\to\infty}\frac{x+2}{2x} = \frac{1}{2}$$

$$\lim_{x\to\infty}\frac{\sqrt{x^2+4}}{2x} = \frac{1}{2}$$

$$\frac{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}}{2x} \qquad \frac{\frac{\sqrt{x^2+4}}{x}}{\frac{2x}{x}}$$

Ex: Find the equation of the vertical and horizontal lines of the following curve:

$$\boxed{4x^2+y^2-8x+8y+16=0}$$



sol: $\frac{dy}{dx}=y'$ $\quad \frac{d}{dx}y^2 = 2y\cdot y'$

$$8x+2y y'-8+8y'+0=0.$$

$$8x-8+y'(2y+8)=0$$

$$(2y+8)\cdot y' = 8-8x \Rightarrow y' = \frac{8-8x}{2y+8} = \frac{4-4x}{y+4}$$

$$\boxed{y' = \frac{4(1-x)}{y+4}}$$

$$\frac{4(1-x)}{y+4}=0 \Leftrightarrow 1-x=0 \text{ and } y+4\neq 0.$$

$$1-x=0 = x=1.$$

$$4\cdot 1^2+y^2-8\cdot 1+8\cdot y+16=0$$

$$y^2+8y+12=0 \Rightarrow y=-2 \text{ or } -6$$

$$y^2+8y+12=(y+2)(y+6)$$

$$\boxed{(1,-2) \text{ and } (1,-6)}$$

Vertical tangent line: $y+4=0$

$\Downarrow$

$y=-4.$

$y' = \dfrac{4(1-x)}{y+4}$

$4x^2 + \cancel{16} - 8x - \cancel{32} + \cancel{16} = 0$

$4x(x-2) = 0 \Rightarrow x = 0$ or $2.$

$\boxed{(0,-4) \text{ and } (2,-4)}.$

logarithmic differentiation

$$\boxed{f(x) = g(x)^{h(x)}}$$

$$\frac{d}{dx} f = ? \qquad \boxed{\frac{d}{dx} x^{(\cos x)}}$$

$$\ln f = \ln(g^h) = h(\ln g)$$

$$\frac{d}{dx}(\ln f) = \frac{d}{dx}(h \cdot \ln g)$$

$$\frac{1}{f} \cdot f' = h' \cdot \ln g + h \cdot \frac{1}{g} \cdot g'$$

$$\Rightarrow f' = f \cdot \left(h' \ln g + h \cdot \frac{g'}{g} \right)$$

$$= g^h \cdot h' \cdot \ln g + g^h \cdot h \cdot \frac{g'}{g}$$

$$\boxed{f' = g^h \cdot h' \cdot \ln g + h \cdot g^{h-1} \cdot g'}$$

$$\boxed{f = g^h}$$

$$\frac{d}{dx}\big(\underbrace{x^{\cos x}}_{}\big) \quad = f$$

$$\boxed{\ln f = \cos x \cdot \ln x}$$

$$\boxed{\frac{f'}{f}} = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$$\Rightarrow f' = x^{\cos x} \cdot \left[-\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}\right]$$

$$= -x^{\cos x} \cdot \sin x \cdot \ln x + x^{(\cos x - 1)} \cos x$$