Ex: $\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}\cdot x^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$\sqrt{x} = x^{\frac{1}{2}}$

$x \to u \to y$

$\frac{d}{dx}\underbrace{\sqrt{\underbrace{x^2+1}_{=u}}}_{=y}$ $\qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{d}{dx}\sqrt{u(x)} = \left(\frac{d}{du}\sqrt{u}\right)\cdot\frac{du}{dx} = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$

$\frac{d}{du}\sqrt{u} = \frac{\cancel{1}}{\cancel{2}}\cdot\frac{1}{\sqrt{u}}$, $\frac{du}{dx} = \cancel{2}x$

Ex: $y = \cos\left(\underbrace{x^{\frac{1}{3}}}_{=u}\right) = \cos(u)$ $\qquad \boxed{\frac{dy}{du} = -\sin(u)}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\sin u\cdot\frac{1}{3}x^{\frac{1}{3}-1} = -\frac{1}{3}\sin(x^{\frac{1}{3}})\cdot x^{-\frac{2}{3}}$

Ex: $y = e^{(2x^2+1)}$, $(2x^2+1) = u$ $\qquad \frac{du}{dx} = \frac{d}{dx}(2x^2+1) = 4x + \underbrace{\frac{d}{dx}1}_{=0}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = e^u\cdot(4x) = e^{2x^2+1}\cdot(4x) = 4x\cdot e^{2x^2+1}$

Ex: $y = \ln(\underbrace{\sin(2x)}_{=u})$ $\qquad \frac{du}{dx} = \frac{d}{dx}\sin(\underbrace{2x}_{=v}) = \frac{d}{dv}\sin(v)\cdot\frac{dv}{dx}$

$= \cos(v)\cdot 2$

$= \cos(2x)\cdot 2$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cos(2x)\cdot 2 = \frac{2\cos(2x)}{\sin(2x)}$

$\left(x \to \underset{}{v} \to u \to y \right)$, $v = 2x$, $u = \sin(v)$, $y = \ln(u)$ $\qquad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$

Ex: $y = \ln\left(\underbrace{\frac{x}{2x+1}}_{u}\right)$

$$\frac{du}{dx} = \frac{1\cdot(2x+1) - x\cdot 2}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\,\frac{1}{(2x+1)^2}$$

$$= \frac{\cancel{2x+1}}{x}\cdot\frac{1}{(2x+1)^{\cancel{2}}} = \frac{1}{x(2x+1)}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$y = \ln\left(\frac{x}{2x+1}\right) = \ln x - \ln(2x+1)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2}{2x+1} = \frac{2x+1-2x}{x(2x+1)} = \frac{1}{x(2x+1)}$$

Ex: $y = \cos^3(\sqrt{x-2}) = \underbrace{\left(\cos(\sqrt{x-2})\right)}_{u}{}^3 = u^3$

$$\boxed{u = \cos(\underbrace{\sqrt{x-2}}_{v})} = \cos(v)$$

$$\frac{dy}{du} = \frac{d}{du}u^3 = 3\cdot u^{3-1} = 3\cdot u^2$$

$$\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx} = -\sin(v)\cdot\frac{1}{2\sqrt{x-2}}$$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 3u^2\cdot(-\sin(v))\frac{1}{2\sqrt{x-2}}\\ &= 3\left(\cos\sqrt{x-2}\right)^2\cdot\left(-\sin(\sqrt{x-2})\right)\frac{1}{2\sqrt{x-2}}\\ &= -\frac{3}{2}\,\frac{\cos^2(\sqrt{x-2})\cdot\sin(\sqrt{x-2})}{\sqrt{x-2}}\end{aligned}}$$

$$\frac{dv}{dx} = \frac{d}{dx}\sqrt{\underbrace{x-2}_{w}}$$

$$= \left(\frac{d}{dw}\sqrt{w}\right)\cdot\frac{dw}{dx}$$

$$= \frac{1}{2}w^{\frac{1}{2}-1}\cdot 1 = \frac{1}{2\sqrt{w}} = \frac{1}{2\sqrt{x-2}}$$

$$w = x-2 \qquad \frac{dw}{dx} = \frac{d}{dx}x - \frac{d}{dx}2$$

$$= 1$$

$$y = \cos^3\left(\sqrt{x-2}\right)$$

$$= \left(\cos\sqrt{x-2}\right)^3$$

$$= 3\left(\cos\sqrt{x-2}\right)^2 \cdot \frac{d}{dx}\cos\left(\sqrt{x-2}\right)$$

$$= 3\left(\cos^2\sqrt{x-2}\right)\left(-\sin\left(\sqrt{x-2}\right)\right)\cdot\frac{d}{dx}\sqrt{x-2}$$

$$= 3\cdot\cos^2\sqrt{x-2}\cdot\left(-\sin\sqrt{x-2}\right)\frac{1}{2}(\ )^{-\frac{1}{2}}\cdot\frac{d}{dx}(x-2)$$

$$= -\frac{3}{2}\cdot\cos^2\sqrt{x-2}\cdot\sin\left(\sqrt{x-2}\right)\frac{1}{\sqrt{x-2}}$$

Ex: $y = \cos(\ln(\sin^2(x)))$

$$\frac{dy}{dx} = -\sin(\ln(\sin^2 x)) \cdot \frac{1}{\sin^2 x} \cdot 2 \cdot \sin(x) \cdot \cos(x) \cdot$$

$$x \xrightarrow{\sin} \sin x \xrightarrow{\cdot^2} \sin^2 x \xrightarrow{\ln} \ln(\sin^2 x) \xrightarrow{\cos} \cos(\ln(\sin^2 x)) = y$$

$y = f(x)$ $\quad \frac{df}{dx} = f' = \frac{dy}{dx}$

$$\frac{df}{dx} = \frac{d}{dx} f$$

$$f'' = \frac{d}{dx} f' = \frac{d}{dx}\frac{d}{dx} f$$

$$= \frac{d^2}{dx^2} f$$

$$f''' = \frac{d}{dx} f'' = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx} f$$

$$= \frac{d^3}{dx^3} f$$

$$\frac{d^2}{dx^2} y = \frac{d}{dx} y'$$

Ex: $y = e^{\overbrace{4x^{-1}}^{u}}$, $y'' = ?$

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx} = e^u \cdot 4(-1) \cdot x^{-1-1}$$

$$= -4 \cdot x^{-2} \cdot e^{4x^{-1}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-4x^{-2} \cdot e^{4 \cdot x^{-1}}\right)$$

$$\frac{d^2y}{dx^2} = y''$$

$$= \frac{d}{dx}\left(-4 \cdot x^{-2}\right) e^{4 \cdot x^{-1}} + \left(-4x^{-2}\right) \frac{d}{dx} e^{4 \cdot x^{-1}}$$

$$= 8 \cdot x^{-3} e^{4 \cdot x^{-1}} - 4x^{-2} \cdot \left(-4x^{-2}\right) e^{4 \cdot x^{-1}}$$

$$= 8 \cdot x^{-3} \cdot e^{4 \cdot x^{-1}} + 16x^{-4} \cdot e^{4x^{-1}}$$

$$= e^{4x^{-1}} \cdot 8 \cdot x^{-4} \cdot (x+2)$$

Implicit differentiation.

$y = f(x)$ explicit function.

$$y^2 + x^2 = 1 \Rightarrow y^2 = 1 - x^2$$
$$\Downarrow$$
$$\boxed{y = \pm\sqrt{1-x^2}}$$

- $y^3 - y^2 x + 1 = 0 \Rightarrow y = ?$

- $y - \cos(y+x) = 0 \Rightarrow y = ?$

$F(x,y) = 0 \Rightarrow$ define y implicitly as a function of x.

$$y^2 + x^2 = 1$$

$$\parallel$$

$$y(x)^2 + x^2 = 1$$

$$\frac{d}{dx}\left(y(x)^2 + x^2\right) = \frac{d}{dx}1 = 0$$

$$\parallel$$

$$\frac{d}{dx}(y(x))^2 + \frac{d}{dx}x^2 \Rightarrow 2y \cdot y' + 2x = 0$$

$$2 \cdot y \cdot y' \qquad \parallel \qquad 2x$$

$$y \cdot y' = -x$$

$$\Downarrow$$

$$\boxed{y' = -\frac{x}{y}}$$