Ex: $$\lim_{x\to+\infty}\frac{x^4-x^3+1}{x^3+2}=\lim_{x\to+\infty}\frac{\frac{x^4-x^3+1}{x^3}}{\frac{x^3+2}{x^3}}=\lim_{x\to+\infty}\frac{x-1+\frac{1}{x^3}}{1+\frac{2}{x^3}}$$

$$=\frac{+\infty-1+\overbrace{\frac{1}{(+\infty)^3}}^{=0}}{1+\underbrace{\frac{2}{(+\infty)^3}}_{=0}}=\frac{+\infty}{1}=+\infty$$

Key: divide by the highest order term of the denominator.

Ex: $$\lim_{x\to+\infty}\frac{x^3-1}{2x^3-\sqrt{x^6-1}}=\frac{\frac{x^3-1}{x^3}}{\frac{2x^3}{x^3}-\frac{\sqrt{x^6-1}}{x^3}}=\frac{1-\frac{1}{x^3}}{2-\sqrt{\frac{x^6-1}{x^6}}}$$

$x^3=\sqrt{x^6}$

$\sqrt{x^6}=x^3$

$$=\lim_{x\to+\infty}\frac{1-\frac{1}{x^3}}{2-\sqrt{1-\frac{1}{x^6}}}=\frac{1-\frac{1}{\infty}}{2-\sqrt{1-\frac{1}{\infty}}}=\frac{1-0}{2-\sqrt{1}}=\frac{1}{1}=1$$

$$\lim_{x\to-\infty}\frac{x^3-1}{2x^3-\sqrt{x^6-1}}=\frac{\frac{x^3-1}{x^3}}{\frac{2x^3}{x^3}-\frac{\sqrt{x^6-1}}{x^3}}=\frac{1-\frac{1}{x^3}}{2-\frac{\sqrt{x^6-1}}{-\sqrt{x^6}}}$$

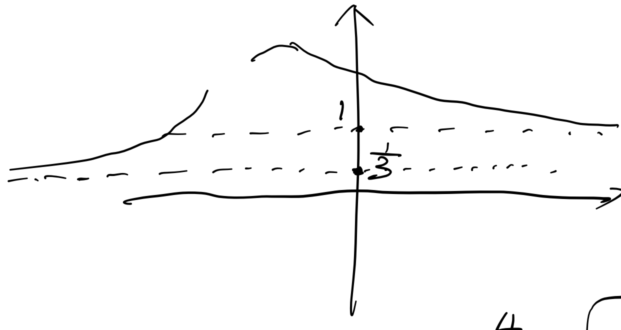
$x<0$

$x^3=-\sqrt{x^6}$

$(-1)^3 \overset{?}{=} \sqrt{(-1)^6}$

$-1\neq\sqrt{1}$

$$=\lim_{x\to-\infty}\frac{1-\frac{1}{x^3}}{2+\sqrt{1-\frac{1}{x^6}}}=\frac{1-\frac{1}{(-\infty)^3}}{2+\sqrt{1-\frac{1}{(-\infty)^3}}}=\frac{1-0}{2+\sqrt{1-0}}=\frac{1}{3}$$



$x^4=\sqrt{x^6}$

Ex: $$\lim_{x\to+\infty}\frac{x^3-1}{x^3-\sqrt{x^6-1}}=\lim_{x\to+\infty}\frac{1-\frac{1}{x^3}}{1-\sqrt{1-\frac{1}{x^6}}}$$

$$=$$

$$+\infty=\frac{1}{0^+}=\frac{1-0}{1-1}=\frac{1-\frac{1}{+\infty}}{1-\sqrt{1-\underbrace{\frac{1}{(+\infty)^6}}_{0}}}$$

$$0<1-\sqrt{\underbrace{1-\frac{1}{x^6}}_{\wedge\atop 1}}\xrightarrow{x\to+\infty}0^+$$

$$0>1-\sqrt{\underbrace{1+\frac{1}{x^6}}_{\vee\atop 1}}\xrightarrow{x\to+\infty}0^-$$

$$\frac{(x^3-1)(x^3+\sqrt{x^6-1})}{(x^3-\sqrt{x^6-1})(x^3+\sqrt{x^6-1})}=\frac{(x^3-1)(x^3+\sqrt{x^6-1})}{1}$$

$$\downarrow x\to+\infty$$

$$+\infty\cdot+\infty=+\infty$$

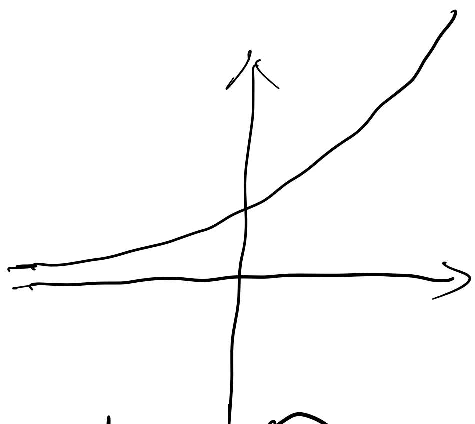
Ex: $$\lim_{x\to-\infty} \frac{e^{2x}+e^{x}}{3\cdot e^{2x}-e^{x}}$$

$$=\lim_{x\to-\infty} \frac{(e^{x})^2+e^{x}}{3\cdot (e^{x})^2-e^{x}}$$

$$=\lim_{x\to-\infty} \frac{e^{x}+1}{3\cdot e^{x}-1}$$

$$=\frac{e^{-\infty}+1}{3\cdot e^{-\infty}-1}$$

$$=\frac{0+1}{3\cdot 0-1}=-1.$$



$$\lim_{x\to-\infty} e^{x}=0$$

$x\to-\infty$

$$\underset{0^2}{\underset{\downarrow}{(e^{x})^2}} \ll \underset{0}{\underset{\downarrow}{e^{x}}}$$

$$\lim_{x\to+\infty} \frac{e^{2x}+e^{x}}{3e^{2x}-e^{x}}$$

$x\to+\infty$

$$=$$

$(e^{x})^{2} \gg e^{x}$

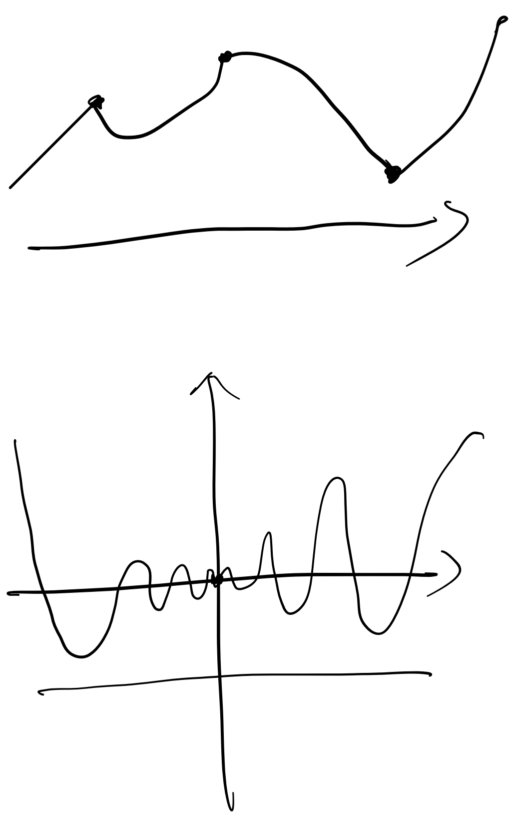
$$\lim_{x\to+\infty} \frac{1+e^{-x}}{3-e^{-x}}$$

$\frac{e^{x}}{e^{2x}} = e^{-x}$

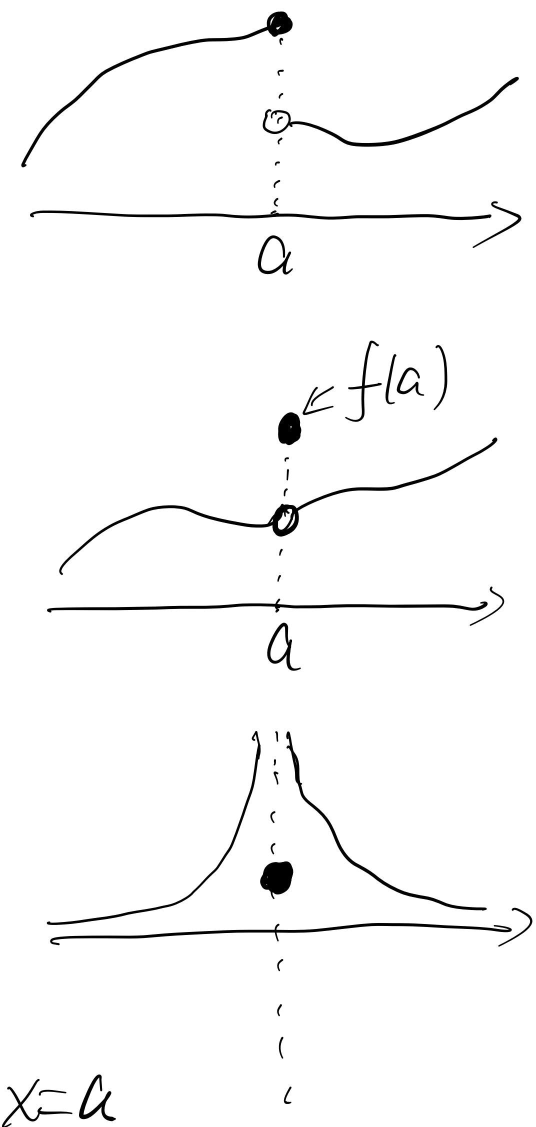
$$=$$

$$\frac{1+e^{-\infty}}{3-e^{-\infty}} = \frac{1+0}{3-0} = \frac{1}{3}$$

- Continuity of a function



not continuous:



Def: f is continuous <u>at $x=a$</u> if the following 3 conditions hold true:

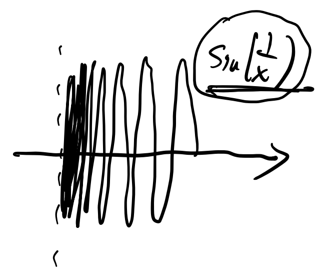
(1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ ~~exosts~~ as a finite number

(3) $\lim_{x \to a} f(x) = f(a)$

Ex: $f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ b & x = 0 \end{cases}$

$\frac{\sin x}{x} \xrightarrow{x \to 0} 1$

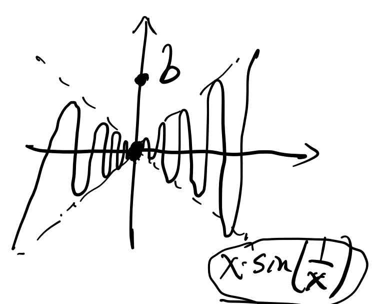


$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

$$-|x| \leq x \cdot \sin\left(\tfrac{1}{x}\right) \leq |x| \Longleftrightarrow \left|x \cdot \sin\left(\tfrac{1}{x}\right)\right| \leq |x|$$

$\left|x \cdot \sin\left(\tfrac{1}{x}\right)\right| = |x| \cdot \left|\sin \tfrac{1}{x}\right|$

$-|x| \to 0$, $\quad x \cdot \sin\left(\tfrac{1}{x}\right) \to 0$, $\quad |x| \to 0$

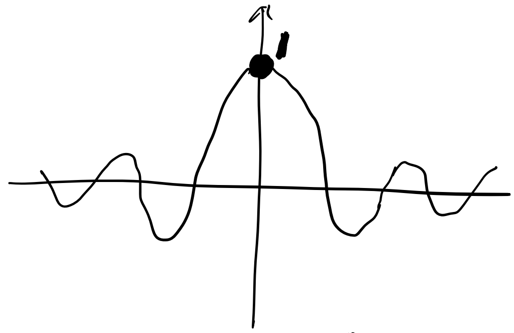


Only when $b=0$ is f continuous.

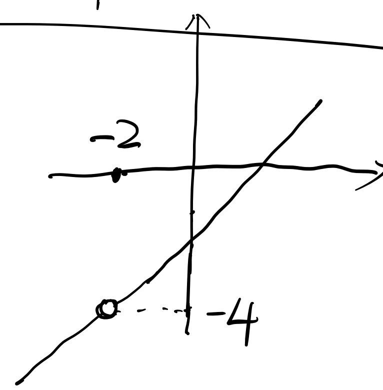
$$\left|x^3 \cdot \sin\left(\frac{1}{x^{100}}\right)\right| = |x|^3 \cdot \left|\sin\left(\frac{1}{x^{100}}\right)\right| \leq |x|^3$$

$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$$f(x) = \begin{cases} \dfrac{x^2-4}{x+2} & x \neq -2 \\ -4 & x = 2 \end{cases}$$



$$\lim_{x \to -2} \frac{x^2-4}{x+2} = \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = \lim_{x \to -2} (x-2) = -4$$

Fact: Rational functions $\dfrac{P(x)}{Q(x)}$

Algebraic functions $\sqrt{P(x)}$

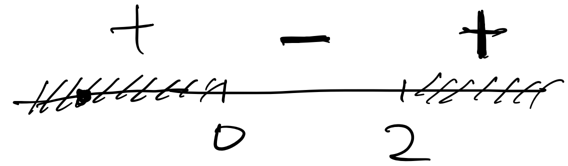
Trigonometric functions

exponential, Logarithmic

Compositions of these are continuous at any point in their domains.

$$f(x) = \sqrt{x^2-2x} = \sqrt{x\cdot(x-2)}$$

Domain of $f = (-\infty, 0] \cup [2, +\infty)$



Ex: $$f(x) = \begin{cases} ax - a^2 & x \geq 3 \\ 2 & x < 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = 2 = \lim_{x\to 3^+} f(x) = 3a - a^2$$

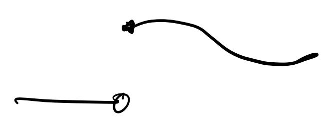
$$2 = 3a - a^2$$

$$\Updownarrow$$

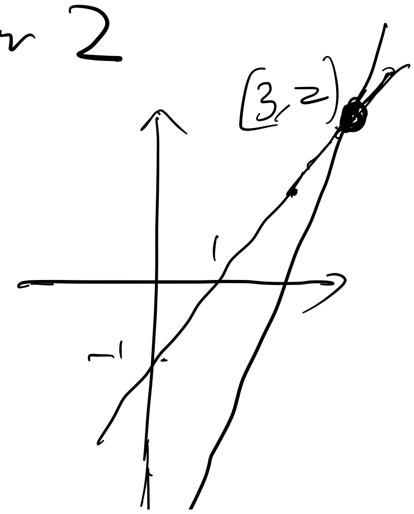
$$a^2 - 3a + 2 = 0$$

$$\parallel$$

$$(a-1)(a-2) \Rightarrow a = 1 \text{ or } 2$$



$a=1$: $$f(x) = \begin{cases} x-1 & x \geq 3 \\ 2 & x < 3 \end{cases}$$



$$a=2: \quad f(x)=\begin{cases} 2x-4 & x\geq 3 \\ 2 & x<3 \end{cases}$$