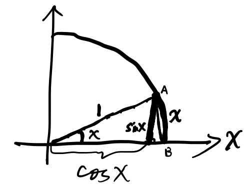
squeeze thm: $f(x) \le g(x) \le h(x)$

If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$.

Ex $\lim_{x\to 0} \sin x = 0$



$-|x| \le \sin x \le |x|$

$\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$

Ex $\lim_{x\to 0} \cos x = \cos 0 = 1.$

Ex $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ ← slope of the tangent line at $(0,0)$.

$\sin x \le x \le \tan x$ for $x>0$

$\tan x = \frac{\sin x}{\cos x}$

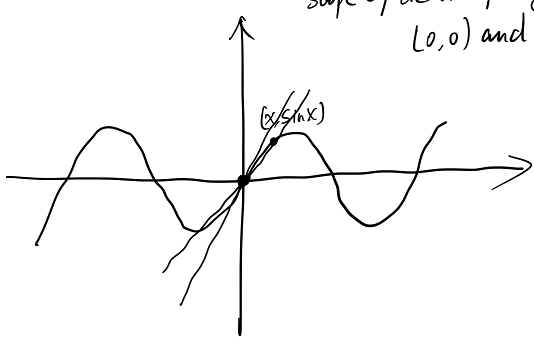
$\Downarrow$

$\boxed{\cos x \le \frac{\sin x}{x} \le 1}$

$\downarrow x\to 0$ $\quad \downarrow \quad \downarrow$

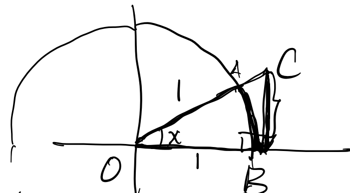
$1 \qquad 1 \qquad 1$

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0}$ ← slope of the line passing through $(0,0)$ and $(x, \sin x)$



$x \le \frac{\sin x}{\cos x}$

$\frac{|CB|}{|OB|} = \tan x$



$\text{Area}(\triangle OBC) = \frac{1}{2} \cdot 1 \cdot \tan x$

$\vee$|

$\text{Area}(\text{sector } OBC) = \pi \cdot \frac{x}{2\pi} = \frac{1}{2} x$

Ex $\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x^2}\right) = 0$

$\left|\sin\left(\frac{1}{x^2}\right)\right| \le 1$

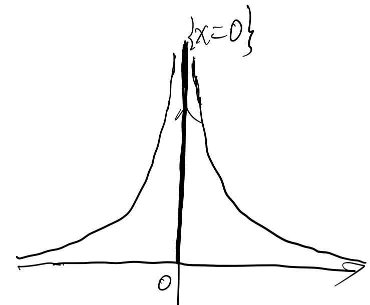
$-|x| \le x \cdot \sin\left(\frac{1}{x^2}\right) \le |x|$

$\downarrow x\to 0 \qquad \downarrow x\to 0$

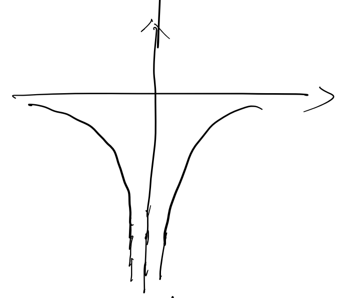
$0 \qquad 0$

Infinite limit

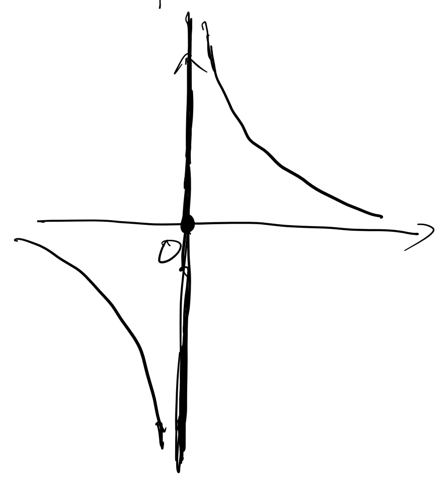
$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$



$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty$$



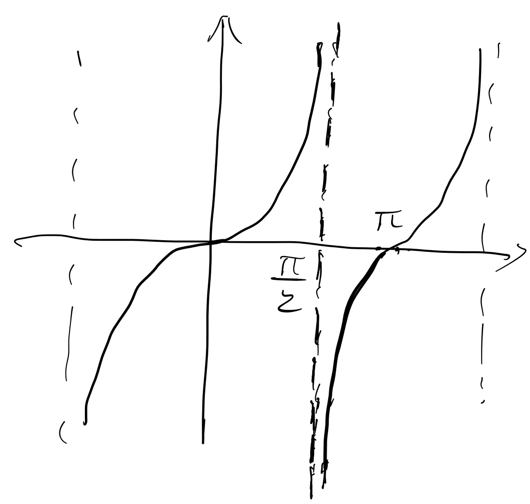
$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE}$$



$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to \frac{\pi}{2}} \tan x$$



$$\lim_{x\to \frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x\to \frac{\pi}{2}^-} \tan x = +\infty$$

$$\lim_{x\to 2} \frac{x-3}{(x-2)^2} = \frac{-1}{0^+} = -\infty$$

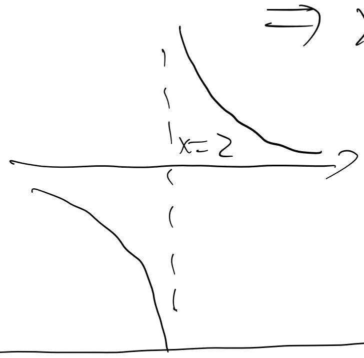
$x-3 \to 2-3 = -1$, $(x-2)^2 \to 0^+$

$$\lim_{x\to 2} \frac{x-2}{(x-2)^2} = \lim_{x\to 2} \frac{1}{x-2} \quad \text{DNE}$$

$x-2 \to 0$

$$\lim_{x\to 2^-} \frac{1}{x-2} = \frac{1}{0^-} = -\infty, \quad \lim_{x\to 2^+} \frac{1}{x-2} = \frac{1}{0^+} = +\infty.$$

$\Rightarrow x=2$ is an asymptote for $f(x) = \frac{x-2}{(x-2)^2}$



Problem: Find vertical asymptotes of $f(x) = \dfrac{x^2-6x+8}{x^2-5x+6}$

Sol. Set $x^2-5x+6=0 \Rightarrow x=2, 3$

$x^2-5x+6 = (x-2)(x-3)$

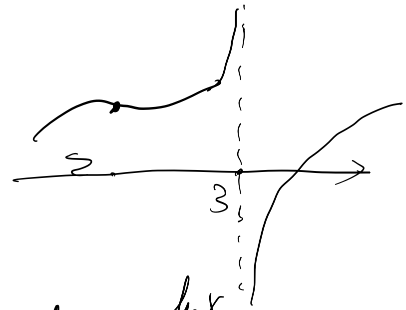
$$\lim_{x\to 2} \frac{x^2-6x+8}{x^2-5x+6} = \lim_{x\to 2} \frac{(x-4)\cancel{(x-2)}}{(x-3)\cancel{(x-2)}} = \frac{2-4}{2-3} = \frac{-2}{-1} = 2.$$

$$\lim_{x\to 3} \frac{x^2-6x+8}{x^2-5x+6} = \lim_{x\to 3} \frac{x-4}{x-3}$$

$$\lim_{x\to 3^-} \frac{x-4}{x-3} = \frac{-1}{0^-} = +\infty$$

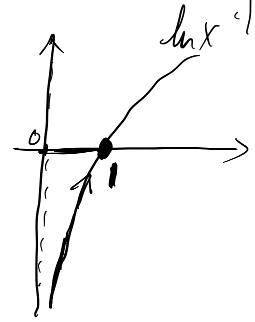
$$\lim_{x\to 3^+} \frac{x-4}{x-3} = \frac{-1}{0^+} = -\infty$$

Ex: $\lim_{x\to -1} \frac{x+2}{\ln(x+2)} = \frac{-1+2}{\ln(1)} = \frac{1}{0}$

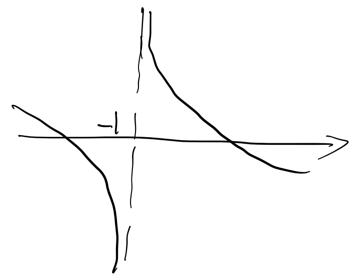


$\lim_{x\to -1^-} \frac{x+2}{\ln(x+2)} = \frac{-1+2}{\ln(1^-)} = \frac{1}{0^-}$

$1^- = -1^- + 2$, $\quad \frac{1}{0^-} = -\infty$

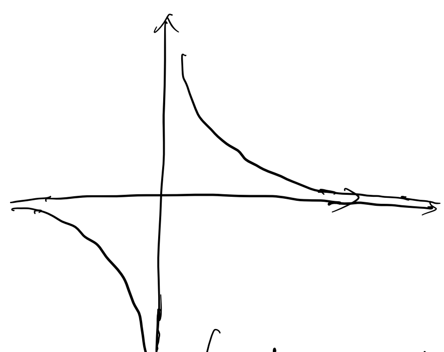


$\lim_{x\to -1^+} \frac{x+2}{\ln(x+2)} = \frac{1}{\ln 1^+} = \frac{1}{0^+} = +\infty$



$\lim_{x\to +\infty} \frac{1}{x} = 0 \quad (0^+)$

$\lim_{x\to -\infty} \frac{1}{x} = 0 \quad (0^-)$



$\lim_{x\to +\infty} \frac{1}{x^2 - x} = \frac{1}{+\infty} = 0$

$\left(\begin{array}{l} \frac{1}{+\infty} = 0^+ \Leftrightarrow \frac{1}{0^+} = +\infty \\ \frac{1}{-\infty} = 0^- \Leftrightarrow \frac{1}{0^-} = -\infty \end{array} \right)$

$x(x-1) = x^2 - x$, $\quad \frac{1}{x^2-x} = \frac{-1}{x} + \frac{1}{x-1} = 0$

$\frac{-1}{x} \xrightarrow{x\to+\infty} \frac{-1}{+\infty} = 0$

$\frac{1}{x-1} \xrightarrow{x\to+\infty} \frac{1}{+\infty} = 0$

$$\lim_{x\to+\infty}\frac{x-2}{x-3}=\frac{+\infty-2}{+\infty-3}=\frac{+\infty}{+\infty}\qquad \lim_{x\to+\infty}\frac{1}{x}=\frac{1}{+\infty}=0$$

$$=\lim_{x\to+\infty}\frac{\frac{x-2}{x}}{\frac{x-3}{x}}=\lim_{x\to+\infty}\frac{1-\frac{2}{x}}{1-\frac{3}{x}}=\frac{1-\lim\limits_{x\to+\infty}\frac{2}{x}}{1-\lim\limits_{x\to+\infty}\frac{3}{x}}=\frac{1-0}{1-0}=1$$

$$\lim_{x\to+\infty}\frac{1x^2-4}{2x^2-5x+1}=\lim_{x\to+\infty}\frac{\frac{x^2-4}{x^2}}{\frac{2x^2-5x+1}{x^2}}=\lim_{x\to+\infty}\frac{1-\frac{4}{x^2}}{2-\frac{5}{x}+\frac{1}{x^2}}$$

$$=\frac{1-\lim\limits_{x\to+\infty}\frac{4}{x^2}}{2-\lim\limits_{x\to+\infty}\frac{5}{x}+\lim\limits_{x\to+\infty}\frac{1}{x^2}}=\frac{1-0}{2-0+0}=\frac{1}{2}$$

$$\boxed{\lim_{x\to+\infty}\frac{a_n x^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_n x^n+b_{n-1}x^{n-1}+\cdots+b_0}=\frac{a_n}{b_n}\qquad b_n\neq 0}$$