Ex: $\lim_{x\to 3} \frac{x^2-6x+9}{x^2-5x+6} \neq \frac{\lim_{x\to 3} x^2-6x+9}{\lim_{x\to 3} x^2-5x+6} =$

$3^2-5\times3+6=0$

$a+b=5$
$a\cdot b=6$

$= \frac{(x-3)^{\cancel{2}}}{(x-2)\cancel{(x-3)}}$

$x^2-5x+6=(x-a)\cdot(x-b)=(x-2)(x-3)$

$\lim_{x\to 3}\frac{x-3}{x-2} = \frac{3-3}{3-2} = \frac{0}{1} = 0.$

$x^2-6x+9=(x-3)\cdot(x-3)=(x-3)^2$

$a+b=6$
$a\cdot b=9$

Ex: $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \neq \frac{\sqrt{3+1}-2}{3-3}$ $\quad \sqrt{3+1}-2=\sqrt{4}-2=2-2=0$ $\quad 3-3=0$

$= \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$

$(\sqrt{a}-\sqrt{b})\cdot(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2=a-b$

$= \frac{x+1-4}{\cancel{(x-3)}(\sqrt{x+1}+2)}$ $\quad x+1-4=\cancel{x-3}$

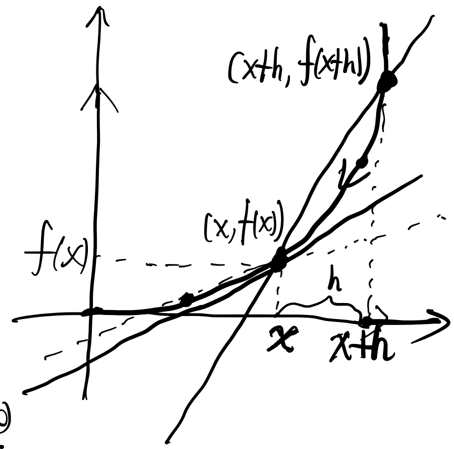
$= \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \frac{1}{4}$

Ex: $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-2} = \frac{\sqrt{3+1}-2}{3-2} = \frac{2-2}{3-2} = 0.$

Ex:

· slope of secant line connecting $(x, f(x))$ and $(x+h, f(x+h))$

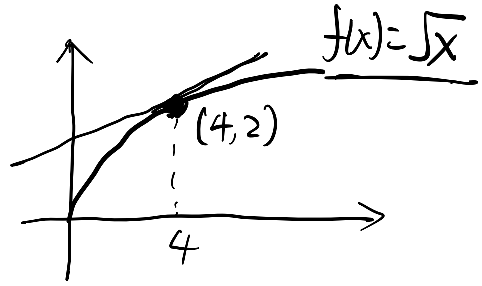
is equal to $\dfrac{f(x+h)-f(x)}{x+h-x} = \dfrac{f(x+h)-f(x)}{h}$




· slope of the tangent line passing $(x, f(x))$

$$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\parallel$$




$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} = \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}$$

$$\parallel$$

$$\lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} = \frac{(\sqrt{4+h})^2-2^2}{h(\sqrt{4+h}+2)} = \frac{\cancel{4}+h\cancel{-4}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$\parallel$$

$$\frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$

Equation for the tangent line passing through $\underset{(x_0,\, y_0)}{(4, 2)}$ :

$$y = \frac{1}{4} x + b$$

$$\frac{y-y_0}{x-x_0} = k$$

$$y = \frac{1}{4}(x-4)+2$$

$$= \frac{1}{4}x - 1 + 2 = \boxed{\frac{1}{4}x+1}$$

$$\frac{1}{4}x+1 = y$$

$$y - y_0 = k(x-x_0)$$

$$\Updownarrow$$

$$\boxed{y = k(x-x_0)+y_0}$$

$$y = kx+b$$

**Ex:** $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x^2}\right)$ $\qquad 0 \cdot \cancel{\sin}\left(\frac{1}{0^2}\right)$

**Squeeze Thm:** $f(x) \le g(x) \le h(x)$

If $\lim\limits_{x\to a} f(x) = L = \lim\limits_{x\to a} h(x)$

then $\lim\limits_{x\to a} g(x) = L$.

$\lim\limits_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ DNE. $\qquad \left|\sin\left(\frac{1}{x^2}\right)\right| \le 1$

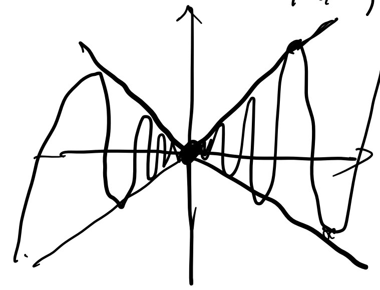
$$\left|x \cdot \sin\left(\frac{1}{x^2}\right)\right| = |x| \cdot \underbrace{\left|\sin\left(\frac{1}{x^2}\right)\right|}_{\le 1} \le |x|$$

$$\left|x \cdot \sin\left(\frac{1}{x^2}\right)\right| = |f(x)|$$

$$\Rightarrow -|x| \le -|f(x)| \le f(x) \le |f(x)| \le |x|$$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$-|x| \leq f(x) \leq |x|$.



$$\lim_{x\to 0} |x| = 0 = \lim_{x\to 0}(-|x|)$$

$$\lim_{x\to 0}(-|x|) = -\lim_{x\to 0}|x|$$

$$\Rightarrow \lim_{x\to 0} x \cdot \sin\left(\frac{1}{x^2}\right) = 0.$$

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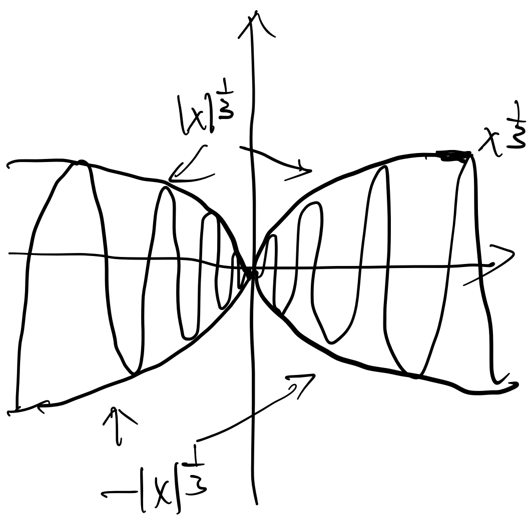
$$\lim_{x\to 0} x^{\frac{1}{3}} \cdot \cos\left(\frac{1}{x}\right) = 0 \qquad \left|\cos\left(\frac{1}{x}\right)\right| \leq 1$$




$$\underline{\left|x^{\frac{1}{3}} \cdot \cos\left(\frac{1}{x}\right)\right| \leq |x|^{\frac{1}{3}}}$$

$$-|x|^{\frac{1}{3}} \leq x^{\frac{1}{3}} \cos\left(\frac{1}{x}\right) \leq |x|^{\frac{1}{3}}$$

$$\downarrow x\to 0 \qquad \downarrow x\to 0 \qquad \downarrow x\to 0$$

$$0 \qquad 0 \qquad 0$$

algebraic functions: $x^{\frac{1}{3}}$, $\sqrt{x^2+2} = f\circ g(x)$

$g(x) = x^2+2$, $f(x) = \sqrt{x}$

Ex: $\lim\limits_{x\to 0} \sqrt{x}$ DNE.

$$\lim_{x\to 0^+} \sqrt{x} = 0 \neq \lim_{x\to 0^-} \sqrt{x} \text{ DNE}$$

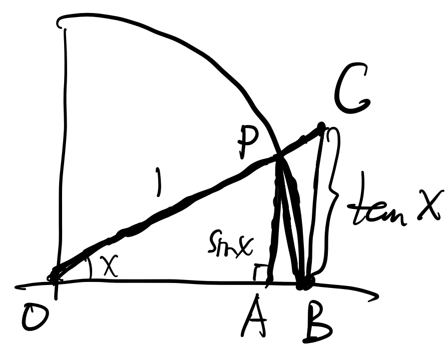
Ex. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Fact: For even functions
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

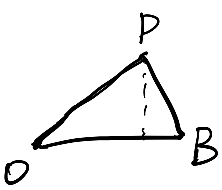
$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$\parallel$$

$$\lim_{x\to 0^-} \frac{\sin x}{x}$$

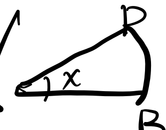



$$\sin x \leq x$$




$$\text{Area}(\Delta_{OPB}) = \frac{1}{2}|OB|\cdot|AP| = \frac{1}{2}\sin x$$

$$\wedge|$$

Area(  ) $= \frac{1}{2} x$

$$\wedge|$$

$$\text{Area}(\Delta_{OBC}) = \frac{1}{2}\cdot|OB|\cdot|BC| = \frac{1}{2}\tan x$$

$$\boxed{\sin x \le x \le \tan x = \frac{\sin x}{\cos x}}$$

$$\Downarrow$$

$$1 \xleftarrow{x \to 0} \cos x \le \frac{\sin x}{x} \le 1$$

$$\lim_{x \to 0} \cos x = \cos 0 = 1 = \lim_{x \to 0} 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$