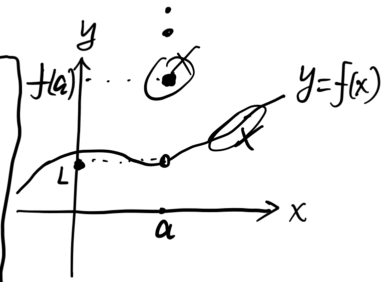
$$\lim_{x\to a} f(x) = L$$

f(x) becomes arbitrarily close to L

for all x that is sufficiently close to a

but not a



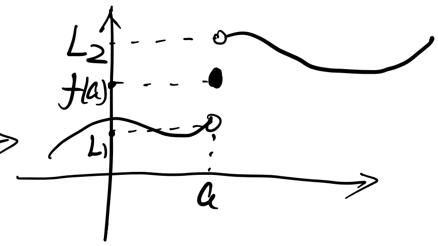

In general, this limit does not depend on f(a).

- $\lim_{x\to a^-} f(x) = L$ if f(x) becomes arbitrarily close to L for all x that is sufficiently close to a and $< a$

(left-sided limit)

- $\lim_{x\to a^+} f(x) = L$ . . .

$\lim_{x\to a^-} f(x) = L_1$, $\lim_{x\to a^+} f(x) = L_2$ $\longrightarrow$



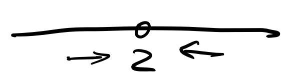

$L_1 \neq L_2 \Rightarrow \lim_{x\to a} f(x)$ does not exist.

- $\lim_{x\to a} f(x)$ exist $\iff$ $\lim_{x\to a^-} f(x) = L_-$, $\lim_{x\to a^+} f(x) = L_+$ exist and $L_1 = L_2$

Ex: $f(x) = \dfrac{x^2-5x+6}{|x-2|}$ $\qquad \lim_{x\to 2} f(x)$ ?

$$|x| = \begin{cases} -x & x<0 \\ x & x\geq 0 \end{cases}$$

$$= \begin{cases} \dfrac{x^2-5x+6}{-(x-2)} & x<2 \\[2ex] \dfrac{x^2-5x+6}{x-2} & x>2 \end{cases}$$



$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} - \frac{x^2-5x+6}{x-2} = \lim_{x\to 2^-} - \frac{\cancel{(x-2)}\cdot(x-3)}{\cancel{x-2}}$$

$x^2-5x+6 = (x-2)(x-3)$ $\quad = \lim_{x\to 2^-} \underline{-(x-3)}$

$\parallel$

$(x-a)\cdot(x-b) = x^2 - \underline{(a+b)}\cdot x + a\cdot b$ $\quad = \lim_{x\to 2^-} (3-x) = 3 - \underset{\parallel\ 2}{\lim_{x\to 2^-} x}$

$$\begin{cases} a\cdot b = 6 \\ a+b=5 \end{cases}$$

$$\boxed{\lim_{x\to 2^-} f(x) = 3-2 = 1}$$

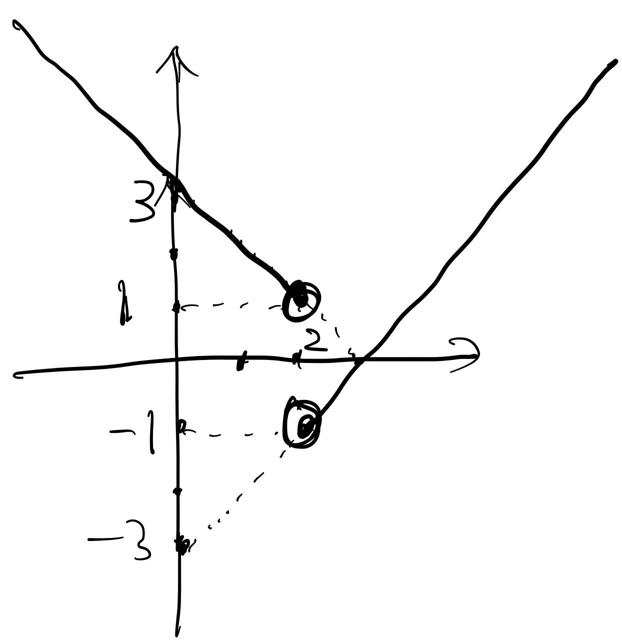
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-5x+6}{x-2} = \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x-3)}{\cancel{x-2}} = \lim_{x\to 2^+} (x-3)$$

$$\boxed{\lim_{x\to 2^+} f(x) = 2-3 = -1}$$

The limit does not exist since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

(<u>DNE</u>)

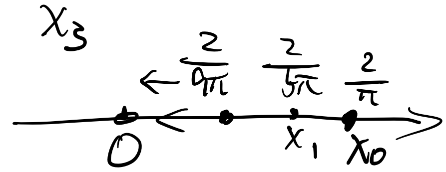
$$f(x) = \begin{cases} \underline{-x+3} & x<2 \\ x-3 & x>2 \end{cases}$$



$$\frac{x_{n+1}-x_n}{x_n \cdot x_{n+1}} = \frac{1}{x_n} - \frac{1}{x_{n+1}} = 2\pi \Longleftrightarrow \boxed{x_{n+1}-x_n = 2\pi\cdot(x_n \cdot x_{n+1})}$$

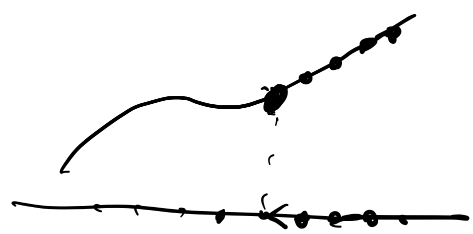
Ex: $\begin{cases} f(x) = \sin\left(\frac{1}{x}\right), & x \neq 0 \\ f(0) = 10000, & x = 0 \end{cases}$ $\quad \lim_{x \to 0} f(x)$ ?

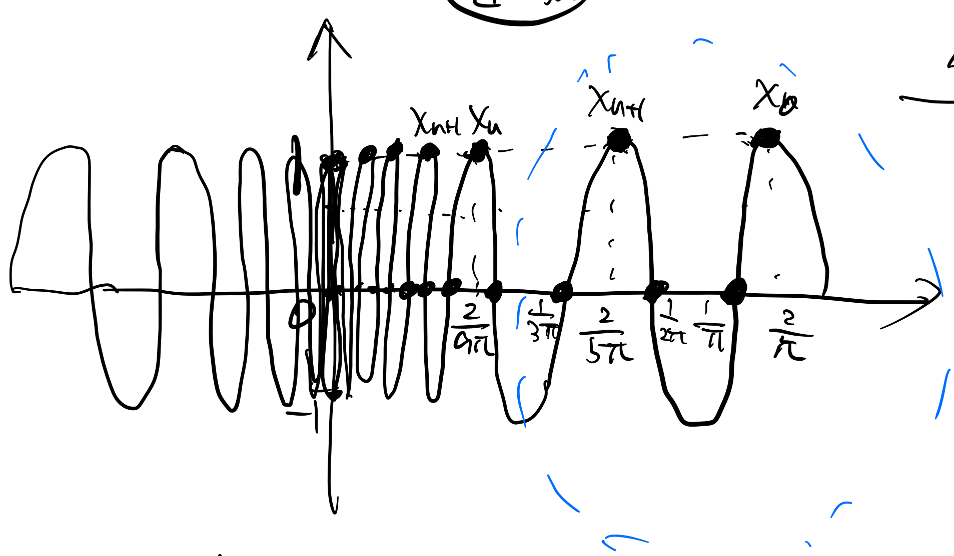
$$x_n = \underset{x_0}{\frac{2}{\pi}},\ \underset{x_1}{\frac{2}{5\pi}},\ \underset{x_2}{\frac{2}{9\pi}},\ \underset{x_3}{\frac{2}{13\pi}},\ \cdots,\ \frac{2}{(4n+1)\pi},$$



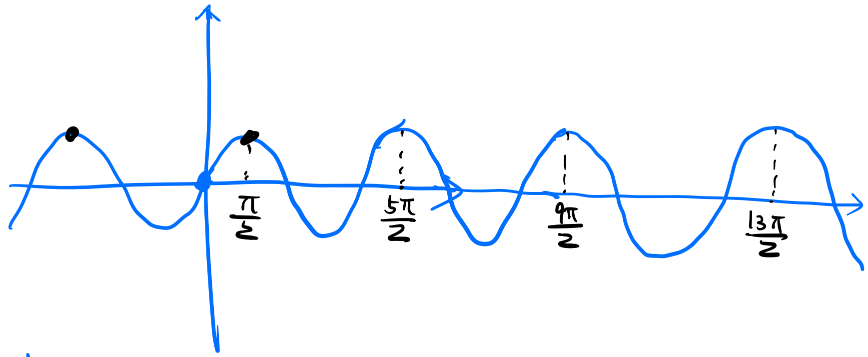

$$\underset{\sin\left(\frac{1}{x_n}\right)}{f(x_n)} = \sin\left(\underset{\frac{1}{\frac{2}{(4n+1)\pi}}}{\frac{(4n+1)\pi}{2}}\right) = \sin\left(n\cdot 2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$\frac{2}{(4n+1)\pi} = x_n$



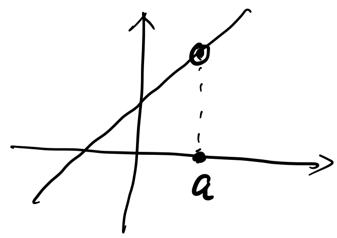



The limit does not exist.




$$\lim_{x\to 0} \sin(x) = 0$$

- $f(x) = k\cdot x + b$




$$\lim_{x\to a} f(x) = ka + b = f(a)$$

- $$\lim_{x\to a} \left(f(x) \pm g(x)\right) = \underbrace{\lim_{x\to a} f(x)}_{= L_1} \pm \underbrace{\lim_{x\to a} g(x)}_{= L_2}$$ if the limits on the rhs exist. (rhs = right-hand-side.)

$$\lim_{x\to a} \left(f(x)\cdot g(x)\right) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$ if the limits on the rhs exist

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ if the limits on the rhs exist and $\lim_{x\to a} g(x) \neq 0$.

Ex: $$\lim_{x\to 2} \frac{2x^3-3}{x+5} = \frac{\lim_{x\to 2}(2x^3-3)}{\lim_{x\to 2}(x+5)} = \frac{2\cdot 2^3-3}{2+5} = \frac{16-3}{7} = \frac{13}{7}$$

Ex: $$\lim_{x\to 2}(2x^3-3) = \lim_{x\to 2} 2x^3 - \underbrace{\lim_{x\to 2} 3}_{= 3} = 16-3 = 13$$

$$\lim_{x\to 2}(2x^3) = \lim_{x\to 2} 2 \cdot \lim_{x\to 2} x \cdot \lim_{x\to 2} x \cdot \lim_{x\to 2} x$$

$$\underset{2\cdot x\cdot x\cdot x}{\shortparallel} \qquad = 2\cdot 2\cdot 2\cdot 2 = 16$$

- For polynomial function $P(x) = a_d\cdot x^d + a_{d-1}\cdot x^{d-1} + \cdots + a_1 x + a_0$

$$\boxed{\lim_{x\to a} P(x) = P(a)}$$

$$\lim_{x\to 3}(5\cdot x^2) = 5\cdot 3^2 = 5\times 9 = 45$$

$$\shortparallel \quad \lim_{x\to 3} 5 \cdot \lim_{x\to 3} x \cdot \lim_{x\to 3} x$$

$$\underset{5}{\shortparallel} \qquad \underset{3}{\shortparallel} \qquad \underset{3}{\shortparallel}$$

- Rational function $\dfrac{P(x)}{Q(x)} = \dfrac{\text{Polyn.}}{\text{polyn.}}$

$$\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{true only if } Q(a)\neq 0.$$

Ex: $\dfrac{x^{100}-5\cdot x^2+4}{x^3-5x+3} = f(x)$. $\quad \lim_{x\to 1} f(x) = \dfrac{1^{100}-5\cdot 1^2+4}{1^3-5+3} = \dfrac{5}{-1} = -5$.

Ex. $\lim_{x\to 1}\dfrac{x^2-1}{x-1} \neq \dfrac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x-1)} = \dfrac{1^2-1}{1-1} = \dfrac{0}{0}$

$$\shortparallel \quad \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$\lim_{x\to 1}(x+1) = 1+1 = 2$$

- $$\lim_{x\to a}\left(c\cdot f(x)\right) = c\cdot \lim_{x\to a} f(x)$$
$$= \lim_{x\to a} c \cdot \lim_{x\to a} f(x)$$

- $$\lim_{x\to a} f(x)^{\frac{1}{n}} = \left(\lim_{x\to a} f(x)\right)^{\frac{1}{n}}$$ if the right hand side is defined

$$\lim_{x\to 2}\left(2x^3+1\right)^{\frac{1}{2}}$$ $\left((-1)^{\frac{1}{2}}\text{ not defined}\right)$

$$= \left(\lim_{x\to 2}\left(2x^3+1\right)\right)^{\frac{1}{2}} = \left(2\cdot 2^3+1\right)^{\frac{1}{2}} = \sqrt{17}$$