Inverse function $\quad f: D \to B$

$f^{-1}: B \to D$ satisfies $\quad f^{-1}(f(x)) = x$ for any $x \in D$

$f(f^{-1}(y)) = y$ for any $y \in B$

Graph of $f = \{(x, f(x)) : x \in D\} = \{(x,y) : y = f(x)\}$

Graph of $f^{-1} = \{(y, f^{-1}(y)) : y \in B\} = \{(x, f^{-1}(x)) : x \in B\} = \{(x,y) : x = f(y)\}$

Graph of $f \xrightarrow{\text{reflect w.r.t. } y=x}$ Graph of f^{-1}

- Inverse of exponential function $=$ logarithmic function

$f(x) = b^x : \mathbb{R} \to \mathbb{R}_{>0}$

$f^{-1}(x) = \log_b x : \mathbb{R}_{>0} \to \mathbb{R}$

satisfies

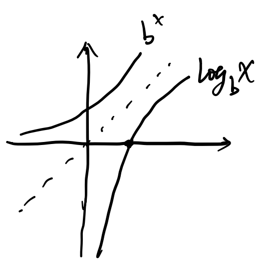
$\log_b b^x = x$ for any $x \in \mathbb{R}$

$b^{\log_b x} = x$ for any $x \in \mathbb{R}_{>0}$

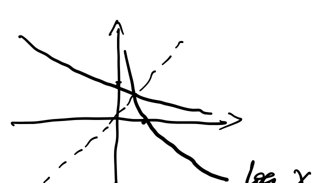
Examples: $\log_2 1024 = \log_2 2^{10} = 10$, $\quad \log_{10} 0.01 = \log_{10} 10^{-2} = -2$

Graphs

$b > 1$



$0 < b < 1$



Properties: $\quad \log_b 1 = 0$, $\quad \log_b (x \cdot y) = \log_b x + \log_b y$, $x, y > 0$.

$\log_b x^c = c \cdot \log_b x$, for any $x > 0$, $c \in \mathbb{R}$.

$\log_b \frac{1}{x} = \log_b x^{-1} = -\log_b x$, $x > 0$.

- $b = e$, $\quad \ln(x) = \log_e x \quad$ natural logarithmic function

- Exponential growth/decay models.

Exponential growth: population growth, epidemic propagation, ...

$$y(t) = y_0 \cdot e^{kt} \qquad y(0) = y_0 \text{ initial data}$$

k: relative growth rate.

Exponential decay: decay of radioactive elements, ...

$$y(t) = y_0 \cdot e^{-\lambda t} \qquad y(0) = y_0 \text{ initial data}$$

λ: relative decay rate

Half-life T satisfies $y(T) = y_0 \cdot \frac{1}{2}$

measurable

$$y_0 \cdot e^{-\lambda T} = y_0 \cdot \frac{1}{2} \Rightarrow e^{-\lambda T} = \frac{1}{2}$$

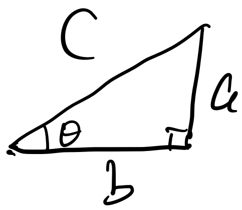
$$\Rightarrow -\lambda T = \ln\frac{1}{2} = -\ln 2 \Rightarrow \lambda = \frac{\ln 2}{T}$$

$$\Rightarrow y(t) = y_0 \cdot e^{-\frac{\ln 2}{T} t}$$

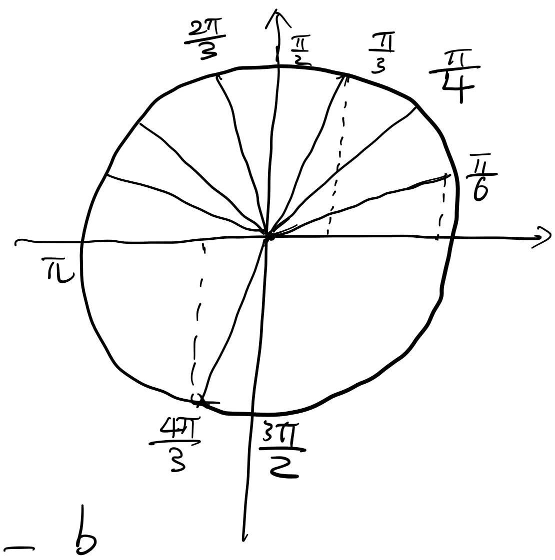
<u>Exercise</u>: $y(t+T) = \frac{1}{2} \cdot y(t)$ for any $t \in \mathbb{R}_{\geq 0}$

Trigonometric functions.

Angle $\theta = \frac{\text{Arclength}}{\text{Radius}}$ (radian)



2π radian $= 360°$ (degree)



$\cos\theta = \frac{b}{c}$, $\sin\theta = \frac{a}{c}$

$\cos^2\theta + \sin^2\theta = \frac{b^2+a^2}{c^2} = 1$

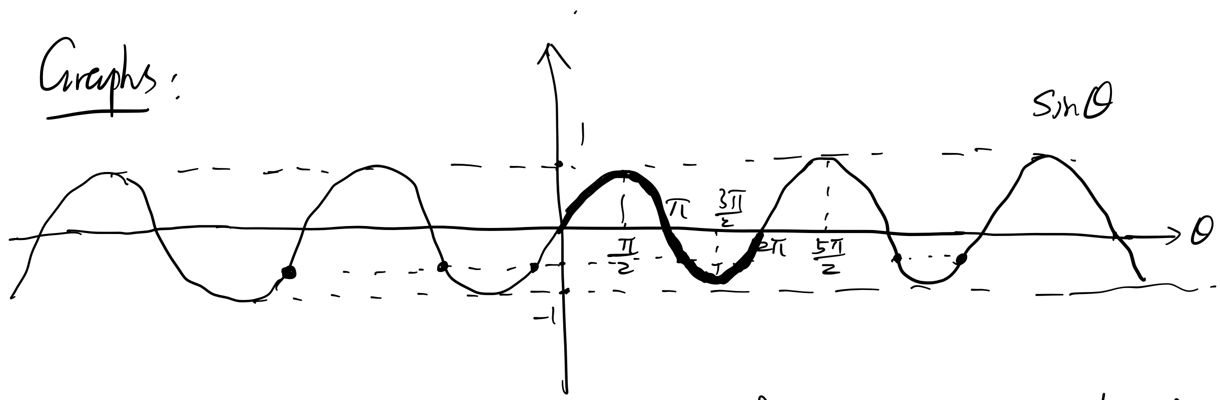
(Pythagorean Theorem)

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{a}{b}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}$

	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\sin\theta$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$	$-\sqrt{3}$
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$

From the circle: $\cos\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\sin\frac{4\pi}{3} = -\frac{1}{2}$

Graphs:



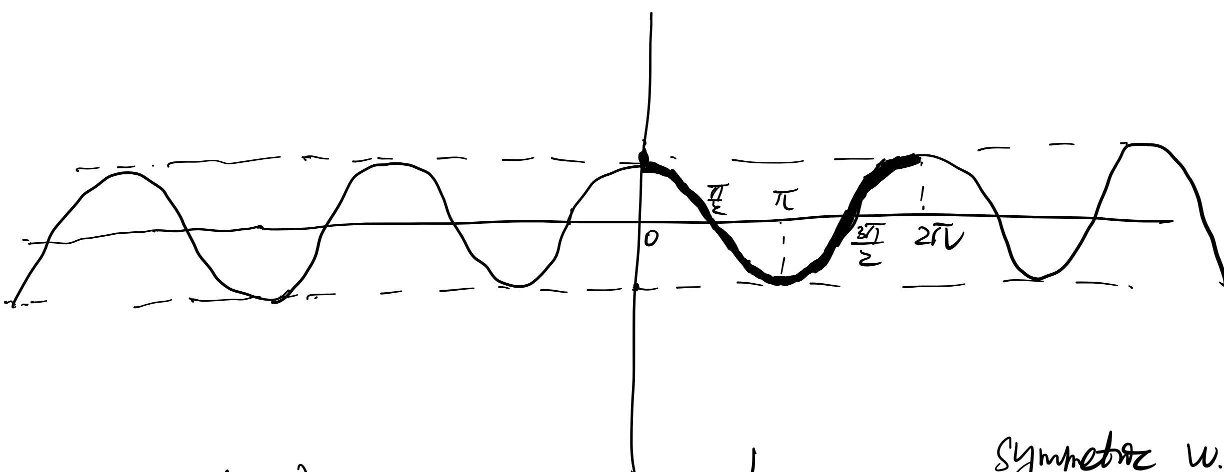
$\sin(-\theta) = -\sin\theta$ odd function. symmetric w.r.t. the origin

$\sin(2\pi+\theta) = \sin\theta$ periodic function of period 2π.

Ex: $\sin\theta = -\frac{1}{2}$,

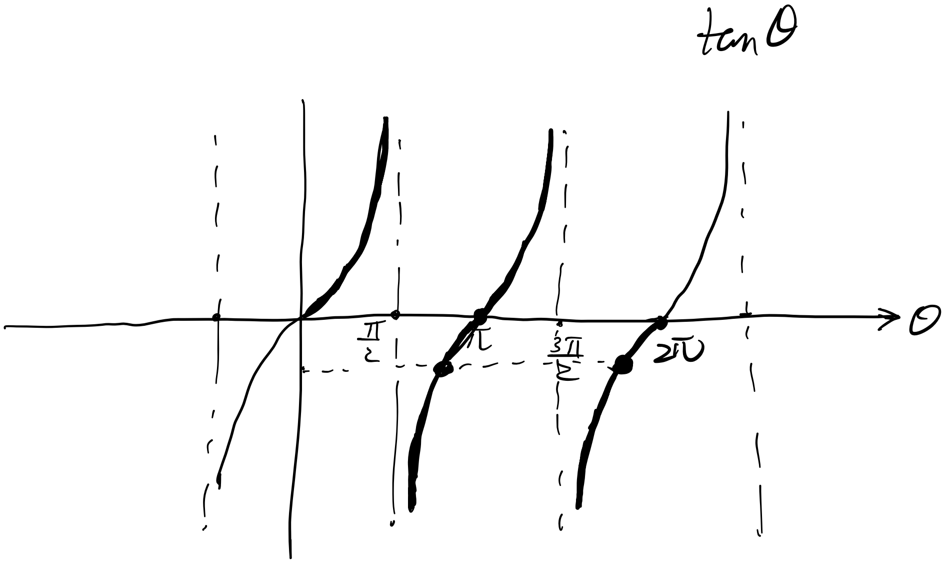
Inside $[0, 2\pi)$: $\theta = \pi + \frac{\pi}{6} = \frac{5\pi}{6}$, $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

All solutions: $\frac{5\pi}{6} + k\cdot 2\pi$, $\frac{11\pi}{6} + k\cdot 2\pi$, $k = 0, \pm1, \pm2, \pm3, \ldots$



$\cos(-\theta) = \cos\theta$. even function symmetric w.r.t. the vertical axis.

$\cos(\theta+2\pi) = \cos\theta$ period 2π



$\tan(-\theta) = -\tan\theta$ odd function

$\tan(\pi+\theta) = \tan\theta$ period π

Ex: $\tan\theta = -\frac{1}{\sqrt{3}}$

inside $[0, 2\pi)$: $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$, $2\pi - \frac{\pi}{6} = \frac{11\pi}{6} = \frac{5\pi}{6} + \pi$

All solutions: $\frac{5\pi}{6} + k\cdot\pi$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$