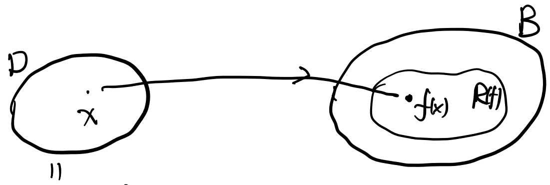
Functions



D = Domain of f = $\mathrm{Dom}(f)$

Range of $f = \{f(x) : x \in \mathrm{Dom}(f)\} \subset B$

$= R(f)$

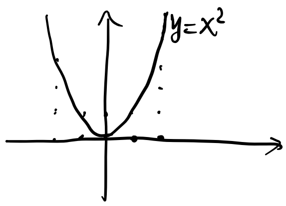
- Graph of f: $\{(x, f(x)) : x \in \mathrm{Dom}(f)\} \subset D \times B$

$= \{(x,y) \in \mathbb{R}^2 : y = f(x)\}$

Ex: $f(x) = x^2$. $\mathrm{Dom}(f) = \mathbb{R} = (-\infty, +\infty)$

$\mathrm{Range}(f) = \mathbb{R}_{\geq 0} = [0, +\infty) =$

Graph



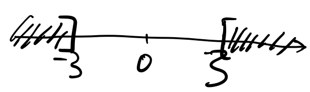
Ex: $g(x) = \sqrt{x^2-9}$ $\mathrm{Dom}(g) = \{x : x^2-9 \geq 0\} = \{x : |x| \geq 3\}$

$\Updownarrow$

$x^2 \geq 9 \Leftrightarrow \sqrt{x^2} \geq \sqrt{9} = 3$

$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\mathrm{Dom}(g) = (-\infty, -3] \cup [3, +\infty)$



$\mathrm{Range}(g) = [0, +\infty)$

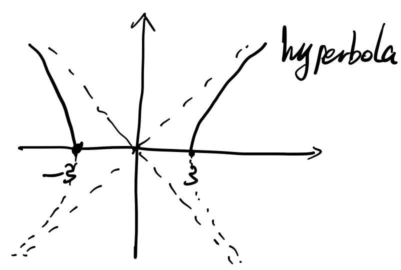
Graph of $g = \{(x,y) : y = \sqrt{x^2-9}\}$

$\Updownarrow$

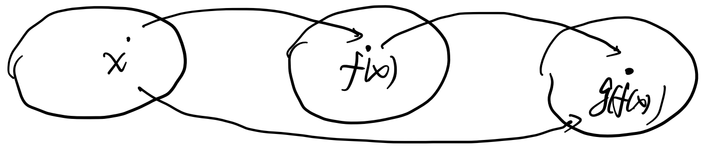
$y^2 = x^2 - 9$ and $|x| \geq 3$, $y \geq 0$

$\Updownarrow$

$x^2 - y^2 = 9$



- Composition of functions:



$g \circ f\,(x)$.

Ex: (continued)

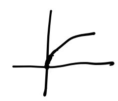
$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{(x^2)^2 - 9} = \sqrt{x^4 - 9}$

$\neq$

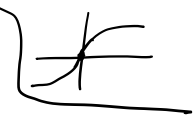
$(f \circ g)(x) = f(g(x)) = f(\sqrt{x^2-9}) = (\sqrt{x^2-9})^2 = x^2 - 9$

In general $g \circ f \neq f \circ g \neq f \times g$

- Class of functions: power functions $x^n = \underbrace{x \cdots x}^{n \text{ copies}}$



root functions $x^{\frac{1}{n}}$ < even root, dom $= [0, +\infty)$; odd root dom $= (-\infty, +\infty)$



exponential functions: Fix $b > 0$. Define b^x for any $x \in \mathbb{R}$

$b^n = b \cdots b$

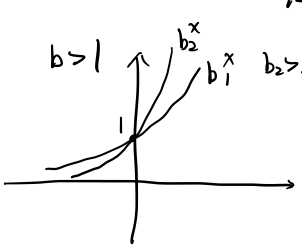
$b^0 = 1$

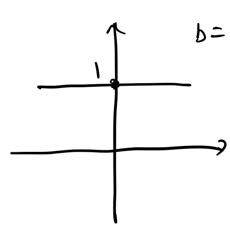
$b^{-n} = (b^n)^{-1} = (b^{-1})^n = \frac{1}{b} \cdots \frac{1}{b}$

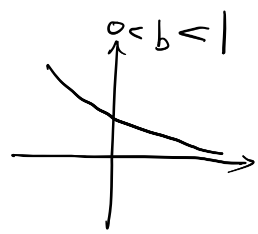
Fact: rational numbers are dense inside $\mathbb{R}$.

$b^{\frac{p}{q}} = (b^p)^{\frac{1}{q}}$

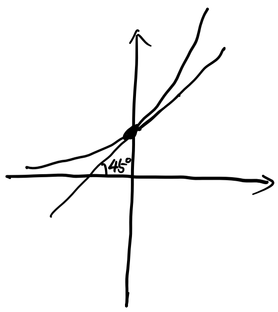
$b^x = \lim_{i \to +\infty} b^{\frac{p_i}{q_i}}$ for x irrational and $\frac{p_i}{q_i} \xrightarrow{i \to +\infty} x$.







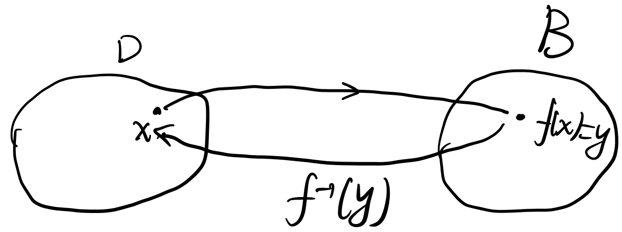
$\underline{b = e}$:
≈ 2.718



slope at $(0,1) = 1 \rightsquigarrow e$ Euler

one of the most important constant in math
compare with $\pi = 3.14159$

- Inverse function:



$f^{-1}: B \to D$ such that

$\cdot\ f^{-1}(f(x)) = x$ for any $x \in D$

and $\cdot\ f(f^{-1}(y)) = y$ for any $y \in B$

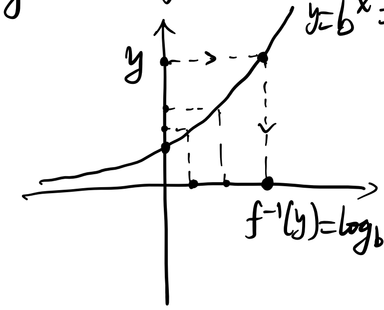
Inverse exists if and only if f is both $\overbrace{\underline{\text{injective}} \text{ and } \underline{\text{surjective}}}^{\text{bijective}}$

injective: different x's give different values

surjective: range fills up the target set

- Logarithmic functions

$y = b^x = f(x) : \mathbb{R} \to \mathbb{R}_{>0} = (0, +\infty)$ bijective



$f^{-1}(y) = \log_b y : \mathbb{R}_{>0} \to \mathbb{R}$ s.t.

$b^{\log_b y} = y$ and $\log_b b^x = x$.

for all $y > 0$ for any $x \in \mathbb{R}$.