Research Statement

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Contents

1 Overview

My research lies at the intersection of algebraic geometry, differential geometry, and several complex variables. The following are some main topics and results of my research projects.

- Systematic application of Minimal Model Program to study K-stability (joint with Chenyang Xu). New valuative criteria for (reduced)-K-(semi)stability of Fano varieties.
- Identification of the second largest volume of n-dimensional Kähler-Einstein manifolds which is obtained only by the quadric hypersurface and the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ (joint with Minghao Miao). This in the toric case answers a conjecture of Andreasson-Berman about arithmetic heights of toric Fano varieties.
- New singularity invariant (normalized volumes) and its application to study optimal degenerations of klt singularities. It leads to an algebraic characterization of unique metric tangent cones on singular KE varieties (with Chenyang Xu and Xiaowei Wang).
- Uniform version of Yau-Tian-Donaldson conjecture for canonical Kähler metrics on Fano varieties, including Kähler-Einstein metrics (joint with Feng Wang and Gang Tian), and weighted Kähler-Ricci solitons (joint with Jiyuan Han). Similar results are obtained for Ricci-flat Kähler cone metrics.

- Construction of algebraic compactification the moduli space of smooth Kähler-Einstein manifolds. As application, we proved the quasi-projectivity of the moduli space. (joint with Xiaowei Wang and Chenyang Xu)
- Existence part of the model version of YTD conjecture which provides the first sufficient (and also necessary) condition for the existence of constant scalar curvature Kähler metrics on polarized manifolds.
- Proof of a conjecture about Kähler compactifications of $\mathbb{C}^n$ and a Liouville type theorem for complex asymptotically conical Calabi-Yau metrics on $\mathbb{C}^3$ (joint with Zhengyi Zhou). We also proved sharp upper bound for the minimal log discrepancy of isolated Fano cones by establishing its connection to orbifold rational curves.
- New Hodge theoretical results: pure Hodge structures for non-Kähler Clemens manifolds, and a converse to Blanchard-Deligne's result about degeneration of Leray spectral sequences for holomorphic submersions.

In the following, I will give more detailed explanation of my works.

2 K-stability of Fano varieties

2.1 Special degenerations and valuative criteria

K-stability was introduced by G. Tian and S. Donaldson to study the existence of canonical Kähler metrics on a polarized projective variety. It is a Hilbert-Mumford type criterion, which is defined in terms of test configurations and CM weights. A test configuration is a flat $\mathbb{C}^*$-equivariant degeneration of a polarized polarized variety (X, L). To each test configuration is attached a weight which is a $\mathbb{C}^*$-weight of a CM line bundle over the Hilbert scheme and which generalizes the classical Futaki invariant associated to holomorphic vector fields.

For Fano varieties, we discovered in [LX14] that K-stability of Fano varieties is "compatible" with the Minimal Model Program (MMP) in the sense that the CM weight is decreasing along a relative MMP. Interestingly, this can be seen a non-Archimedean analogue of the fact that the Mabuchi functional (whose Euler-Lagrangian equation is the Kähler-Einstein metric) is decreasing along the Kähler-Ricci flow. Starting with any test configuration, the MMP performs various surgeries to "simplify" the total space and the central fibre of the degeneration, ends with a nice one called special degeneration. Tian used special degeneration to define K-stability because of the corresponding compactness result in metric geometry, while Donaldson used more general test configurations. As a consequence of the above monotonicity, we proved the equivalence of Tian and Donaldson's definition of K-stability for all Fano varieties.

Boucksom-Hisamoto-Jonsson associated to any special test configuration of a $\mathbb{Q}$-Fano variety X a special divisorial valuation ord_F on its function field $\mathbb{C}(X)$ (for a prime divisor on some birational model $\mu : Y \to X$). In [Li17b] I realized that the CM weight associated to a special test configuration can be expressed in the following simple form:

$$\beta(F) := A_X(\mathrm{ord}_F) - \frac{1}{(-K_X)^n} \int_0^{+\infty} \mathrm{vol}(\mu^*(-K_X) - x \cdot \mathrm{ord}_F) dx \tag{1}$$

where $A_X(\mathrm{ord}_F) = 1 + \mathrm{ord}_F(K_Y - \mu^* K_X)$ is the log discrepancy of F, an important invariant in birational algebraic geometry. As a consequence, I obtained the following valuative criteria for K-(semi)stability of Fano varieties:

Theorem 1 ([Li17b, Li22a]).
- *A Fano variety X is K-semistable if and only if $\beta_X(F) \geq 0$ for any divisorial valuation ord_F on $\mathbb{C}(X)$. Moreover, if $\beta_X(F) > 0$ for any divisorial valuation ord_F then X is K-stable.*
- *Let $\mathbb{G}$ be a connected reductive subgroup of $\mathrm{Aut}(X)_0$. Then X is uniformly $\mathbb{G}$-stable if and only if there exists $\delta > 1$ such that for any divisorial valuation $v := \mathrm{ord}_F$ over X, there exists an ξ in the Lie algebra of the center torus of $\mathbb{G}$ such that $A_X(v_\xi) \geq \delta \cdot S_{-K_X}(v_\xi)$.*

Here v_ξ is a twisted of the valuation v introduced in [Li22a]. K. Fujita independently obtained these criteria. With Jiyuan Han, we also generalized the criteria to the more general setting of weighted K-stability [HL24]. Such valuative criterion has lead to many interesting development including Fujita-Odaka's δ-invariant and Abban-Zhuang's method of testing K-stability. It also inspires a similar valuative criterion for generalized polarized manifold (see section 4.2).

Optimal degenerations of Fano varieties Motivated by the study of Kähler-Ricci flow and my work on normalized volumes (section 3.1), I (jointly with Jiyuan Han) introduced a non-linear version of the β-invariant ((1)) and a new minimization problem. For any $\mathbb{Q}$-Fano variety, denote by Val_X the space of real valuations on the function field $\mathbb{C}(X)$. For any $v \in \mathrm{Val}_X$, define:

$$\tilde{\beta}(v) = A_X(v) + \log\left(\frac{1}{(-K_X)^{\cdot n}} \int_{\mathbb{R}} e^{-t}(-d\mathrm{vol}(\mathcal{F}_v^{(t)})\right). \tag{2}$$

Theorem 2 ([HL24]). *For any $\mathbb{Q}$-Fano variety, there exists a minimizer v_* of $\tilde{\beta}$ in Val_X which is quasi-monomial (equivalently $\mathrm{rat.rk(v)} + \mathrm{tr.deg.(v)} = \dim X$). Moreover if v_* comes from a special $\mathbb{R}$-degeneration, then v_* is unique such that its special fibre is weighted K-semistable and has a unique degeneration to a weighted K-polystable Fano variety.*

This is an algebraic version of the Hamilton-Tian conjecture about Kähler-Ricci flows on Fano manifolds, and an analogue to the existence/uniqueness of Harder-Narasimhan and Jordan-Hölder filtrations for holomorphic vector bundles. It also generalizes Tian-Zhu's unique result for Kähler-Ricci soliton vector fields, and proves a conjecture by Chen-Sun-Wang on the algebraic uniqueness of normalized Kähler-Ricci flow limit for Fano manifolds. Later Blum-Liu-Xu-Zhuang proved that v_* has a finite generation property and indeed comes from a special $\mathbb{R}$-degeneration.

2.2 Sharp volume bounds and rational curves

A major recent topic of my research, carried out jointly with Minghao Miao, concerns establishing a sharp upper bound for the anticanonical volume of K-semistable Fano manifolds.

Theorem 3 ([LM25]). *If X is a K-semistable Fano manifold of dimension n that is not $\mathbb{P}^n$, then $\mathrm{vol}(X) = (-K_X)^n \leq 2n^n$, with equality if and only if X is isomorphic to either a quadric hypersurface in $\mathbb{P}^{n+1}$ or the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

Previously K. Fujita in [Fuj18] showed that the largest possible anticanonical volume of a K-semistable Fano manifold is $(n+1)^n$ which is achieved only by $\mathbb{P}^n$. He proved it by applying the valuative criterion to the exceptional divisor of standard blow-up at a point. A key innovation in our proof is the use of *valuations arising from weighted blowups along minimal rational curves*, which gives a satisfactory explanation of the previous mysterious fact that there are two Fano manifolds achieving the second largest volume.

A minimal rational curve is an immersion $f : \mathbb{P}^1 \to X$ such that

$$f^*TX \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(d-2)} \oplus \mathcal{O}^{\oplus n-d+1}.$$

By the theory of rational curves on Fano manifolds, such curves exist and sweep out X. There were important classification of Fano manifolds with $d \in \{n, n+1\}$ based on work of Mori and Miyaoka ([CD15]), which is the relatively easier case for establishing the volume bound. We considered weighted blowup with weights $(1^{\oplus d-2}, 2^{\oplus n-d+1})$ along $f(\mathbb{P}^1)$ adapted to the above splitting, and applied the valuative criterion to arrive at a delicate inequality relating their log discrepancies to volumes, the analysis of which yields the desired bound for the difficult cases $2 \leq d \leq n-1$. We are trying to study similar bounds for Fano orbifolds, the interest of which will be explained in section 3.2.

One important application of Theorem 3 is the answer to a question of Andreasson-Berman in [AB24], who defined a canonical arithmetic height for arithmetic Fano varieties defined over $\mathbb{Z}$. They first proved that the canonical height of an arithmetic Fano variety $\mathcal{X}_{\mathbb{Z}}$ is finite if and only if its complexification X is K-semistable. This latter fact depends on my result in [Li17a] which states that a Fano variety is K-semistable if and only if the associated Ding functional defined on the space of continuous positively curved metrics on $-K_X$ is bounded from below.

Conjecture 1 (Andreasson-Berman)**.** *Let $\mathcal{X}_{\mathbb{Z}}$ be an arithmetic normal Fano variety of dimension n over $\mathbb{Z}$. If the complexification X is K-semistable, then the following height inequality holds: $(\overline{-K_{\mathcal{X}}})^{n+1} \leq (\overline{-K_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ where $-K_{\mathbb{P}^n_{\mathbb{C}}}$ is endowed with the volume normalized Fubini-Study metric. Moreover, the equality holds if and only if $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and the metric is Fubini-Study.*

They reduced the conjecture in the toric case to the conjecture that the volume of K-semistable toric Fano variety is that is not $\mathbb{CP}^n$ bounded from above by $\mathrm{vol}(\mathbb{P}^1 \times \mathbb{P}^{n-1}) = 2n^n$. So Theorem 3 answered their conjecture in the toric case. For general K-semistable Fano varieties, it is an open question how to reduce the sharp arithmetic height bounds to sharp volume bounds like the result in Theorem 3.

3 Singularity invariants

3.1 Normalized volume

Motivated by the study of metric tangent cones and results from Sasaki geometry, I introduced a new invariant for klt (Kawamata-log-terminal) singularities. Denote the space of valuations centered at a klt singularity $x \in X$ by $\mathrm{Val}_x(X)$. The volume of a valuation $v \in \mathrm{val}_x(X)$ is:

$$\mathrm{vol}_{X,x}(v) := \limsup_{m \to +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!},$$

where $\mathfrak{a}_m(v)$ denotes the valuation ideals: $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}$.

Definition 1 ([Li18]). *For a valuation $v \in \mathrm{Val}_x(X)$, we define the normalized volume*

$$\widehat{\mathrm{vol}}_X(v) := \begin{cases} A_X(v)^n \cdot \mathrm{vol}(v) & \text{if } A_X(v) < +\infty \\ +\infty & \text{if } A_X(v) = +\infty. \end{cases}$$

The local volume of $x \in X$ is defined as $\widehat{\mathrm{vol}}(x, X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_X(v)$.

I proved $\widehat{\mathrm{vol}}(x, X) > 0$ ([Li18]) for any n-dimensional klt singularity $x \in X$ and proposed some conjecture which has been studied by several people and completed in recent work of Xu-Zhuang. I continued to work on the connection between volume minimization and K-semistability of $\mathbb{Q}$-Fano varieties, and more generally Fano cones. Moreover together with Chenyang Xu, we studied the behavior of normalized volume under Minimal Model Program, establishing a monotonicity, analogous to the case of global CM weight but in a local setting ([LX20]). These have the some applications:

- Another characterization of K-semistable log-Fano variety: A log Fano variety (X, D) is K-semistable if and only if ord_X is the minimizer of the $\widehat{\mathrm{vol}}_C(v)$ where C is the Fano cone over (X, D) ([Li17b, LL19, LX20]).
- Algebraic characterization of metric tangent cones of Gromov-Hausdorff limits of KE manifolds, answering a conjecture of Donaldson-Sun ([LX18, LWX21]).
- Connection with logarithmic version of Miyaoka-Yau inequalities for K-semistable log Fano pairs ([Li21]).

Liu-Xu proved that $\widehat{\mathrm{vol}}(x, X) \leqslant n^n$, with the equality holds iff $x \in X$ is a smooth point. Spotti-Sun proposed the following conjecture which is a local analogue of the result in Theorem 3.

Conjecture 2 (ODP conjecture). *Let (X, x) be an n-dimensional non-smooth klt singularity. Then $\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n$ with equality holds iff (X, x) is the affine cone over a smooth quadric hypersurface.*

This conjecture has applications to compactification of moduli space of KE manifolds, and would also imply a sharp volume bound for any singular K-semistable Fano variety. This implication follows from a local-to-global inequality proved by Fujita-Liu: for any closed point x on a K-semistable Fano variety,

$$(-K_X)^n \leq \frac{(n+1)^n}{n^n} \widehat{\mathrm{vol}}(x, X).$$

There are further application of normalized volume to controlling minimal log discrepancies, Cartier indices and consequently to bounding singularities. Such boundedness results combined with a general inequality of local-to-global type have been applied to study termination of flips in the Minimal Model Program ([HQZ25]).

One can reduce Conjecture 2 to the special case when (x, X) is a K-semistable isolated Fano cone singularity. For a K-semistable Fano cone singularity (X, x), the associated Fano orbifold $\mathcal{S} = X/\mathbb{C}^*$ is a K-semistable Fano orbifold. Generalizing the work [LM25], by studying the theory of minimal rational curves in the orbifold setting and its relationship with K-semistability, we expect to get good volume estimate as in the smooth case in Theorem 3. To prove conjecture 2, one still needs to control the Fano index in the orbifold setting. In general, the index can be arbitrarily large as can be seen from the examples of weighted projective spaces. However, there is a way to bound a modified Fano index. This is closely related to my work about the minimal log discrepany invariants to be explained next.

3.2 Minimal log discrepancy and orbifold rational curves

Let (x, X) be a germ of n-dimensional klt singularity. Its minimal log discrepancy of (x, X) is defined as:

$$\mathrm{mld}(x, X) = \min\{A_X(E) : \mathrm{cent}_X(E) = x\}.$$

For example, the minimal log discrepancy of a smooth point is equal to n, achieved by the exceptional divisor of the ordinary blowup at the point. Shokurov conjectured that the minimal log discrepancy is lower semicontinuous in $x \in X$, which would imply

Conjecture 3 (Shokurov)**.** *For any klt singularity (x, X), $\mathrm{mld}(x, X) \leq n$ and the equality holds if and only if (X, x) is a smooth point.*

This is known to be true in the following cases: (i) $\dim X \leq 3$; (ii) (X, x) is a local complete intersection; (iii) (X, x) is toric, or quotient singularity; (iv) (X, x) is a cone singularity over a smooth Fano manifold. For the last class, Shokurov's conjecture is true because of the well-known criterion of Kobayashi-Ochiai: an $(n-1)$-dimensional Fano manifold has Fano index n if and only if it is $\mathbb{P}^{n-1}$. In [LZ25], we proved a new result concerning Shokurov's conjecture 3:

Theorem 4 ([LZ25])**.** *Let (X, x) be an isolated Fano cone singularity. Then $\mathrm{mld}(X, x) \leq n$.*

We proved this by establishing a new connection between the mld invariant and the dimension of moduli space of orbifold rational curves. This was initiated in our work [LZ24] where a formula for the mld invariants of isolated Fano cones was derived. For any element g in a stabilizer of an orbifold point, we consider $\mathcal{M}_{(1,g)}(\mathbb{P}^1(1, \ell), \mathcal{S})$, the moduli stack of twisted maps (or good orbifold maps) $f : \mathbb{P}^1(1, \ell) \to \mathcal{S}$ that satisfies $\mathrm{ev}_0(f) \in \mathcal{S}_1$ and $\mathrm{ev}_\infty(f) \in \mathcal{S}_g$. We proved the following key estimate:

$$\mathrm{mld}(x, X) \leq \min_g\{\dim \mathcal{M}_{(1,g)}(\mathbb{P}^1(1, \ell), \mathcal{S}), \ell \in \mathbb{Z}_{>0}\}.$$

Next we used Mori's method in [Mor79] in the orbifold setting. In other words, the idea to get the sharp upper bound of the mld invariant is as follows: if the dimension of $\mathcal{M}_{(1,g)}(\mathbb{P}^1(1, \ell), \mathcal{S})$ is bigger than $n+1$, then we can fix two points $f(0)$ and $f(\infty)$ to bend-and-break the orbifold rational curve into an rational curve with smaller deformation space. Interestingly, the tools developed in the study of orbifold Gromov-Witten invariants (by Abramovich-Graber-Vistoli, Chen-Ruan) play an important role in our degeneration argument. As a consequence, we proved a generalization of Mori's result to the orbifold setting:

Theorem 5 ([LZ25])**.** *For any Fano orbifold $\mathcal{S}$, there exists $\ell \in \mathbb{Z}_{>0}$ and a twisted map f : $\mathbb{P}^1(1, \ell) \to \mathcal{S}$ with $d_{g^{-1}}(f) := -K_{\mathcal{S}} \cdot f_*\mathbb{P}^1(1, \ell) + \mathrm{age}(g^{-1}) \leq n$.*

We expect that this bound on modified Fano index will be useful to prove Conjecture 2. We also reduced the characterization of the equality case in Theorem 12 to the following new conjecture which is an analogue of Mori-Mukai's conjecture in the orbifold setting:

Conjecture 4 ([LZ25]). *Let* (S, Δ) *be an* $(n-1)$*-dimensional Fano orbifold. Assume that* $d_{g^{-1}}(f) \geq n$ *for any orbifold map* $f : \mathbb{P}(1, \ell) \to \mathcal{S}$*. Then* $\mathcal{S}$ *is isomorphic to a finite quotient of a weighted projective space.*

For smooth Fano manifolds, Mori-Mukai's conjecture was proved in [CMSB02]. We are exploring the possibility of developing similar methods for Fano orbifolds. As a preliminary step, we already established an orbifold version of Mori's theorem:

Theorem 6 ([LZ25]). *Let* $\mathcal{S}$ *be a Fano orbifold with an ample orbifold tangent bundle. Then* $\mathcal{S}$ *is isomorphic to a finite quotient of a weighted projective space.*

The idea is similar to Mori's by using the twisted maps $f : \mathbb{P}^1(1, \ell) \to \mathcal{S}$ obtained in Theorem 5 to sweep out the whole orbifold. A key fact that makes this work is that the pull-back orbifold vector bundle $f^*T\mathcal{S}$ splits into orbifold line bundles over $\mathbb{P}^1(1, \ell)$. The argument used for proving Theorem 6 will be essential for our plan to attack Conjecture 4.

4 YTD conjecture and pluripotential theory

4.1 Kähler-Einstein metrics on Fano varieties

YTD conjecture for Fano varieties A long-term theme of my research is to understand the Yau–Tian–Donaldson (YTD) conjecture in full generality, which predicts that, for a polarized projective variety (X, L), the existence of canonical Kähler metrics in $c_1(L)$ is equivalent to suitable (uniform) K-stability condition.

Jointly with Gang Tian and Feng Wang, I proved the uniform version of YTD conjecture for *all* $\mathbb{Q}$-Fano varieties, including highly singular ones.

Theorem 7 ([LTW21, LTW22, Li22a]). *A* $\mathbb{Q}$*-Fano variety admits a KE metric if and only if it is reduced-uniformly K-stable.*

Earlier breakthroughs by Chen-Donaldson-Sun and Tian solved the smooth case. However, new difficulty arises in the singular setting, and analytic tools alone are not sufficient. The key new ingredients to our proof are uniform estimates obtained on resolutions of singularities that depends on the algebraic valuative criterion for (uniform)-K-stability in Theorem 1 and generalization of Berman-Boucksom-Jonsson's non-Archimedean techniques (in [BBJ21]).

Joint with Jiyuan Han, in [HL24] we also extended these results to obtain YTD-type theorems for weighted Kähler–Ricci solitons including Sasaki–Einstein metrics and special extremal metrics, hence unifying the treatment of several formerly disparate class of canonical Kähler/Sasaki metrics (see [ALN25]).

Moduli of smoothable Fano KE varieties In joint work with Xiaowei Wang and Chenyang Xu ([LWX18, LWX19]), we constructed proper algebraic moduli spaces of smoothable (but possibly singular) KE Fano varieties. This is based on analytic results about Gromov-Hausdorff limits and provided the first general construction of compact moduli in the higher-dimensional Fano case. As application, we proved the quasi-projectivity of the moduli space of smooth KE Fano manifolds. A technical ingredient for the latter fact is the continuity of Ding energy proved in [Li17a] for proving the nefness of CM line bundle on the compactification.

More recent work of Liu-Xu-Zhuang ([LXZ22] and others have provided powerful algebraic approach to the construction of proper moduli space of K-polystable Fano varieties and an equivalence between reduced uniform K-stability with K-polystability (and hence proving the original YTD conjecture for all $\mathbb{Q}$-Fano varieties by combining with Theorem 7 and a result of Berman).

4.2 Yau-Tian-Donaldson conjecture for polarized manifold

While the YTD conjecture is nown known for Fano varieties (even singular ones), the case of general polarization is more subtle and remains open at this time. In the work [Li20], I proved the a version of the YTD conjecture for arbitrary polarized manifolds by introducing the notion of *uniform K-stability for all models*. A model $(\mathcal{X}, \mathcal{L})$ is the same as test configuration except we allow $\mathcal{L}$ to be just big instead of semiample. One may define non-Archimedean functionals $\mathbf{M}^{\mathrm{NA}}$ and $\mathbf{J}^{\mathrm{NA}}$ purely in terms of movable intersection numbers of $\mathcal{L}$ and $K_{\mathcal{X}}$ on $\mathcal{X}$. I proved

Theorem 8 ([Li20])**.** *If there exists $\gamma > 0$ such that*

$$\mathbf{M}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geq \gamma \, \mathbf{J}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \quad \textit{for } \text{all} \textit{ models } (\mathcal{X}, \mathcal{L}),$$

then (X, L) admits a cscK metric.

This criterion is purely algebro-geometric, which can be used to prove the uniform version of YTD conjecture for all spherical manifolds (generalizing toric manifolds). Very recent work of Boucksom-Jonsson and Darvas-Zhang showed that the sufficient algebraic condition is also necessary, and hence giving a uniform version of YTD correspondence. The key step in our proof of 8 is to use pluripotential theory to prove the fact that destabilizing geodesic rays can always be approximated by test configurations such that the non-Archimedean energy functionals converge. Connections of this fact with singular plurisubharmonic functions allows me construct a counterexample to Demailly's conjecture about approximation of Monge-Ampère mass ([Li24a]). The result in Theorem 8 also leads to the study of valuative criterion (e.g. [BJ21]) that generalize the criterion in Theorem 1.

A central obstacle to extending the full YTD conjecture to general polarized manifolds is the need to approximate model metrics by genuine test configurations without losing control of non-Archimedean entropy which is essentially a sum of log discrepancies:

$$\mathrm{Ent}^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = K_{\mathcal{X}/X \times \mathbb{C}} \cdot \langle \mathcal{L}^n \rangle = \int_{X^{\mathrm{NA}}} A_X(v) \mathrm{MA}^{\mathrm{NA}}(\phi_{\mathcal{L}}).$$

This challenge led me to study deeper asymptotic invariants of big line bundles, in particular the *first Riemann–Roch coefficient*:

$$r_1(X, L) := \langle L^{d-1} \rangle \cdot K_X,$$

where $\langle L^{d-1} \rangle$ is the positive intersection product of Boucksom–Favre–Jonsson. For a big and nef line bundle, $r_1(X, L)$ appears in the term of order m^{d-1} in the Riemann–Roch expansion for $h^0(X, mL)$. However, for a general big line bundle L, the coefficient $r_1(X, L)$ is far more mysterious and depends on subtle birational positivity properties. I formulated a Fujita-type approximation conjecture for the first Riemann–Roch coefficient.

Conjecture 5 ([Li23a])**.** *There exist birational models $\mu_m : X_m \to X$ and decompositions*

$$\mu_m^* L = A_m + E_m,$$

with A_m ample and E_m effective, such that

$$A_m^d \to \mathrm{vol}(L), \qquad K_{X_m} \cdot A_m^{d-1} \to r_1(X, L).$$

This refines a conjecture of Boucksom-Jonsson, and would strengthen the classical Fujita approximation by extending it beyond the leading term to the next-order asymptotic. It is known to hold for varieties admitting a birational Zariski decomposition (e.g. Mori dream spaces) and for certain Fano-type varieties where $-K_X$ is nef.

5 Other Results in complex geometry

5.1 Compactification of $\mathbb{C}^n$ and Calabi-Yau metrics

Recently jointly with Zhengyi Zhou, I proved some results in complex geometry by connecting it to singularity theory/Fano geometry in algebraic geometry. The first is a solution to a long-standing conjecture of Brenton-Morrow.

Theorem 9 ([LZ24]). *Let X be a smooth Kähler manifold. Assume that Y is a smooth complex submanifold such that $X \setminus Y$ is biholomorphic to $\mathbb{C}^n$. Then X is biholomorphic to $\mathbb{CP}^n$ and Y is a linear subspace.*

Previously, it is known that X^n and Y^{n-1} are Fano manifolds satisfying $-K_X = r[Y]$ with $r > 1$. Combining this with a construction of Tian-Yau lead us to prove a more general result:

Theorem 10 ([LZ24]). *Let $(\mathcal{X}, \mathcal{Y})$ be an orbifold compactification associated to an asymptotically conical Kähler metric on $\mathbb{C}^n$. Then the metric tangent cone C at infinity satisfies $\mathrm{mld}(C) = n$.*

Such type of orbifold compactfication was obtained in my work [Li20] for the regular case and then generalized to the orbifold case by Conlon-Hein. As a corollary, we proved a Liouville type theorem which answers a conjecture of Tian in dimension 3 and reduces the higher dimensional case to Conjecture 3.

Theorem 11 ([LZ24]). *Let ω_g be a complete Ricci-flat Kähler metric on $\mathbb{C}^3$. Assume that ω_g has a maximal volume growth such that the metric tangent cone at infinity has a smooth link. Then ω_g is isometric to the flat metric on $\mathbb{C}^3$.*

Theorem 10 was proved by establishing new formulas for minimal log discrepancy in terms of symplectic invariants (generalized Conley-Zehnder indices and symplectic homology):

Theorem 12 ([LZ24]). *Let $o \in \mathcal{C}$ be an n-dimensional isolated Fano cone singularity. For any conic contact form η on the contact link M with a Liouville filling W, we have:*

$$2 \cdot \mathrm{mld}(o, \mathcal{C}) = \inf_{\gamma} \mu_{\mathrm{CZ}}(\gamma) + n - 1 = \inf\{d; SH_d^{+,S^1}(W; \mathbb{Q}) \neq 0\} + n - 1 \tag{3}$$

where $\mu_{\mathrm{CZ}}(\gamma)$ denotes a Conley-Zehnder index of any closed Reeb orbit γ, and $SH_*^{+,S^1}(W;\mathbb{Q})$ denotes the $\mathbb{Q}$-coefficient S^1-equivariant positive symplectic homology.

This formula is motivated by the work of McLean [McL16] and is proved based on an explicit formula for mld and a Morse-Bott spectral sequence. Combining this formula with functorial properties of symplectic homology, we proved Theorem 9-12 and consequently Conjecture 4. We are trying to extend the formula (3) to more general singularities by developing some ideas from symplectic topology. This may lead to applications to problems about rigidity of singular Calabi-Yau metrics.

5.2 Zoll manifolds with entire Grauert tubes

Another application of Fano geometry is to Zoll manifolds with entire Grauert tubes. A Zoll manifold is a Riemannian manifold (M, g) whose geodesics are all closed, simple and of the same period. By the Bott-Samelson theorem, any simply connected Zoll manifold has the same cohomology ring as one of CROSSes' (i.e. S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{OP}^2$) so that one can talk about types of Zoll manifolds according to cohomology rings. The only other examples of Zoll manifolds are certain exotic Zoll metrics on S^n and it is a widely open question to classify all Zoll manifolds, which is believed to be very limited.

Assume that the Riemannian manifold (M, g) is real analytic. By works of Grauert, Lempert-Szöke and Gullemin-Stenzel etc., there is a canonical complexification of (M, g) which means that there is a unique integrable complex structure on the neighborhood of the zero section of the tangent bundle TM so that the associated geodesic foliation has holomorphic leaves. If the complex structure exists on the whole tangent bundle, then we say that (M, g) has an entire Grauert tube. The CROSSes have entire Grauert tubes. In these cases, the tangent bundle TM are affine manifolds and can be compactified to homogeneous Fano manifolds.

Recently my Ph.D. student Kyobeom Song and I proved the following rigidity result:

Theorem 13 ([LS25])**.** *A Zoll manifold of type $\mathbb{CP}^n$ with an entire Grauert tube is isometric to the standard Fubini-Study metric on $\mathbb{CP}^n$.*

This answers a question of Burns-Leung [BL18], who proved a similar result for the S^n type. To prove Theorem 13, we first used Burns-Leung's work to compactify TM to a complex manifold X with a smooth divisor D, the closed points of which parametrizes oriented closed geodesics. This compactification result is parallel to the compactification result in [Li20]. Our key observation is that the both X and D are Fano manifolds whose Fano indices can be determined by Morse indices of the closed geodesics. This is closely related to our work on minimal log discrepancy which relates mld to Conley-Zehnder indices. Indeed, the Morse indices of closed geodesics are known to be special example of Conley-Zehnder indices in the setting of geodesic flows and the minimal log discrepancy of a regular Fano cone is nothing but the Fano index. So this observation fits well with the result in Theorem 12.

For any Zoll manifold of type $\mathbb{CP}^n$, we showed that X is of dimension $2n$ with second Betti number 2 and Fano index $n+1$. Then we used a classification result of Wiśniewski: If an n-dimensional Fano manifold X with Fano index r satisfies $r \geq \frac{n+2}{2}$, then $\mathrm{Pic}(X) = \mathbb{Z}$ unless X is isomorphic to the product of projective spaces $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

It is a natural problem to use Fano geometry to classify all Zoll manifolds of types $\mathbb{HP}^n$ and $\mathbb{OP}^2$ with entire Grauert tubes. By using Mukai's classification of Fano manifolds with co-index 3, K. Song is able to deal with the case of type $\mathbb{HP}^2$. The difficulty for the remaining cases is that currently there are no characterization of the Fano manifolds $\mathrm{Gr}(2, 2n+2)$ (with $n \geq 3$) and E_6/P_1 in terms of Fano indices. On the other hand, we know that the anti-canonical volumes of Zoll manifolds coincide with the corresponding models. Moreover we will explore the use of rational curves to characterize the two special Fano manifolds. This is meaningful since the compactified leaves of geodesic foliation furnish a large family of rational curves which may have useful geometric information for the characterization.

5.3 Hodge Theoretical results

Polarize Hodge structure of Clemens manifolds Let X be a Calabi-Yau threefold. A conifold transition first contracts X along disjoint rational curves with normal bundles of type $(-1,-1)$, and then smooth the resulting singular complex space $\bar{X}$ to a new compact complex manifold Y. Such Y is called a Clemens manifold and is often non-Kähler. In [Li24b], I proved that any smoothing Y of $\bar{X}$ satisfies $\partial\bar{\partial}$-lemma and that the resulting pure Hodge structure of weight three on $H^3(Y)$ is polarized by the cup product. As a consequence we should be able to study the deformation and degeneration of Clemens manifolds using the variational/degeneration theory of Hodge structures. This is interesting since it is expected that such non-Kähler Calabi-Yau manifolds are important for connecting different moduli spaces of projective Calabi-Yau threefolds (Reid's Fantasy). This question was left unanswered from 1980's and Friedman [Fri19] proved that $\partial\bar{\partial}$-lemma holds outside of a proper real analytic subvariety in a neighborhood of deformation space, while our result covers a whole neighborhood.

My proof refines Friedman's technique and uses Deligne's decomposition of limiting mixed Hodge structure coupled with some elementary linear algebra argument. Even though well-understood in the Kähler setting, this type of argument seems not noticed in the non-Kähler setting. My result has been generalized to other non-Kähler manifolds in several works (by K.W. Chen, T.J. Lee, T. Sano).

A converse to the Blanchard-Deligne theorem For any holomorphic submersion $\pi : X \to B$ of compact complex manifolds, we derive a criterion for X to have Kähler structures. In a classical paper [Bla56], under the assumption that the holomorphic submersion is isotrivial and also satisfies the extra condition that $\pi_1(B)$ acts trivially on $H^1(F)$, Blanchard proved that X is Kähler if and only if B is Kähler and the transgression map $H^1(F) \to H^2(B)$ is 0. I generalized his criterion to all holomorphic submersions:

Theorem 14 ([Li23b])**.** *There is a Kähler metric on X if and only if all of the following conditions are satisfied:*

1. *There is an element $[\omega] \in H^0(B, R^2\pi_*\mathbb{R})$ restricts to be a Kähler class on $X_b = \pi^{-1}(b)$ for any $b \in B$.*
2. *$d_2[\omega] = 0$ where $d_2 : E_2^{0,2} = H^0(B, R^2\pi_*\mathbb{R}) \to E_2^{2,1} = H^2(B, R^1\pi_*\mathbb{R})$ is the differential in the 2nd page of the Leray spectral sequence.*
3. *B is Kähler.*

This can be seen as a Kähler analogue of Thurston's construction of symplectic structures for symplectic fibrations, and also as a converse to the fundamental result of Blanchard and Deligne which states that for a smooth Kähler fibration, the Leray spectral sequence degenerates at the second page E_2. I used the above result to answer some special case of Streets-Tian question about Hermitian-symplectic structures.

The proof of the above theorem uses Deligne's Hodge theory for cohomology groups with coefficients of local systems coming from polarized variation of Hodge structures. I plan to use S. Zucker's work for Hodge theory with degenerating coefficients to extend the above result to the general fibrations with possibly singular fibers.

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