Some ternary homogeneous structures

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1/7, or possibly 7/1

Problem Session, Oberwolfach

¹— BW 😊
Concrete classification problems for ternary languages

- Homogeneous hypertournaments
- Homogeneous families of linear orders

Some examples and some questions.

(Slightly fuller version in "TernaryProblems")
Some ternary homogeneous structures

1. Hypertournaments

2. Families of Linear Orderings
Some ternary homogeneous structures

Hypertournaments

$t$-hypertournaments

**Definition**

$t$-ary, antisymmetric (exactly $\text{Alt}_t$-symmetric).
The classification problem

**Problem**

*Finitely many (boundedly many?) homogeneous $t$-hypertournaments for each $t$?*

Known for $t = 2$ and possible only because there is a unique $t$-type up to symmetry (no Henson construction).

**Problem**

*For $t \geq 3$, are there any infinite ones which are not $(t + 1)$-constrained?*

(One for $t = 2$.)
Some ternary homogeneous structures

Hypertournaments

Even hypertournaments

**Definition**

\( t \) odd:
Parity of a \( t \)-hypertournament on a \((t + 1)\)-set: well-defined (take a linear order and count increasing \( t \)-tuples which are hyperarcs).

**Even**: All parities on \((t + 1)\)-sets are even.

**Proposition**

An even homogeneous \( t \)-hypertournament (so, \( t \) odd) restricts to a homogeneous \((t − 1)\)-hypertournament, and is determined by the latter.
The case $t = 3$: Catalog

There is a unique homogeneous $t$-hypertournament $H_{t+1}$ of order $t + 1$, $\text{Aut}(H_{t+1}) = \text{Alt}_{t+1}$.

Three 4-types: $H_4$, $C_4$ (circular order), $O_4$ (odd).

Proposition

A homogeneous 3-hypertournament has one of the following forms.

- **Finite**, order 1, 2, 4, 8 (trivial; $H_4$; $A(1, \mathbb{F}_8)$ (W. Kantor, 1972).
- **Generic cyclic order**, realizes only type $C_4$.
- **Generic even 3-hypertournament**, omits $O_4$.
- **Realizes** $C_4$, $O_4$, omits $H_4$. [Generic exists—any others?]
- **Realizes all 4-types** [Generic exists—any others?]
1 Hypertournaments

2 Families of Linear Orderings
Some ternary homogeneous structures

Families of Linear Orderings

Homogeneous FLO

**Definition**

FLO: \((A, R(x, y, z))\): \(R\) irreflexive relation; \(R(a, x, y) = <_a\) linear on \(A \setminus \{a\}\) and not derived from a constant order on \(A\).

This seems like a class worth working out the Ramsey theory for, for any examples one can come up with, whether or not one has chances for a real classification. In any case the first step is to work out the “natural” examples and even this is not done systematically.

**Unfortunate Proposition**

There are \(2^{\aleph_0}\) homogeneous FLO.
Control the cyclic part of the relation.

- Build an infinite antichain of irreducible cyclic structures contained in FLOs
- Amalgamate without introducing new irreducible cyclic structures.

Antichain: derived from cyclic orders on at least 5 points by replacing triples $x < y < z$ with $x, y, z$ consecutive by $x, z, y$.

Amalgamation: $A_0 \subseteq A_0 \cup \{a_1\}, A_0 \cup \{a_2\}$: make $A_0 <_{a_1} a_2$ and $A_0 <_{a_2} a_1$. 
Some ternary homogeneous structures

Families of Linear Orderings

So...

Problem

How many homogeneous FLOs are there with trivial cyclic part?

Problem

What exactly are the 4-constrained homogeneous FLO’s?