Genericity, Generosity, and Tori

Gregory Cherlin

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I Structure of connected groups of finite Morley rank
   - with and without 2-tori

II Application: Poizat’s problem on generic equations
   Groups of unipotent type

III Details
   - Relation with Carter subgroups
   - Genericity arguments
     - Limoncello—Degenerate type groups—Toricity

IV Application: Permutation groups
   - Generic $t$-transitivity
Connected groups of finite Morley rank (in general)

- Generic covering and conjugacy theorems
- Definable hulls of $p$-tori
Essential Notions

- **Morley rank** \( \text{rk} (X) \)
- **Generic set**: \( \text{rk} (X) = \text{rk} (G) \)
- **Connected group**
  \[
  [G : H] < \infty \implies G = H.
  \]
  \[
  X, Y \subseteq G \text{ generic} \implies X \cap Y \text{ generic}
  \]
- **\( d(X) \)**: definable subgroup generated by \( X \).
$p$-torus: divisible abelian $p$-group

Types:
- Degenerate: No infinite 2-subgroup
- Even: Nondegenerate, no nontrivial 2-torus ("characteristic two type")

$p$-unipotent: definable, connected, bounded exponent, nilpotent $p$-group
Without 2-tori

\[ 1 \leq O_2(G) \leq G \]

\( O_2(G) \): maximal unipotent 2-subgroup

\[ \tilde{G} = G/O_2(G) \]

\[ \tilde{G} = U_2(\tilde{G}) \rtimes \hat{O}(\tilde{G}) \]
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- \(U_2(\bar{G})\): product of algebraic groups;
- \(\hat{O}(G)\): no involutions
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With 2-tori

**Theorem (G_2)**

The generic element of \( G \) belongs to \( C^\circ(T) \) for a unique maximal 2-torus \( T \).
\[ \bar{G} = G/O_2(G) = U_2(\bar{G}) \ast \hat{O}(\bar{G}) \]

\(U_2(\bar{G})\): product of algebraic groups; \(\hat{O}(G)\): no involutions

**Ingredients**
Without 2-tori

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Ingredients

Theorem (E)

A simple group of even type is algebraic.

Theorem (D)

A connected degenerate type group contains no elements of order two.
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**Ingredients**

**Theorem (E)**

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**Methods:** Finite group theory, good tori, Wagner on fields of finite Morley rank—classification

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Without 2-tori

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Ingredients

Theorem (E)

A simple group of even type is algebraic.

Methods: Finite group theory, good tori, Wagner on fields of finite Morley rank—classification

Theorem (D)

A connected degenerate type group contains no elements of order two.

Methods: Black box group theory, genericity arguments—soft methods
Theorem (E)

A simple group of even type is algebraic.

1st wave: No bad fields, no degenerate type simple sections.
2nd wave: No degenerate type simple sections.
3rd wave: General case (tori)
Theorem (E)

A simple group of even type is algebraic.

Definition

A definable divisible abelian subgroup $T$ of $G$ is a **good torus** if every definable subgroup of $T$ is the definable hull of its torsion subgroup.
**Theorem (E)**

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**Definition**

A definable divisible abelian subgroup $T$ of $G$ is a **good torus** if every definable subgroup of $T$ is the definable hull of its torsion subgroup.

**Rigidity properties:**

- **R-I** $N^\circ(T) = C^\circ(T)$
- **R-II** Any uniformly definable family of subgroups of $T$ is finite.

Theorem (E)

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Corollary (U)

A connected group of finite Morley rank without p-tori has degenerate type.
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Direct proof

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$M = N(U)$. 

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**Strong Embedding:** If $M \cap M^g$ contains an involution then $g \in M$. Hence: All involutions of $U$ are conjugate under the action of $M$. 

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Direct proof

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But $M^o = C^o(U)$ in view of

(a) the absence of $p$-tori;
(b) Wagner’s theorem: the multiplicative group of a field of finite Morley rank in positive characteristic is a good torus;
Corollary (U)

A connected group of finite Morley rank without $p$-tori has degenerate type.

Direct proof

$U$ the connected component of a Sylow 2-subgroup.
$M = N(U)$.
All involutions of $U$ are conjugate under the action of $M$.

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forcing finitely many involutions in $U$. 
Theorem (G_p)

The generic element of G belongs to $C^\circ(T)$ for a unique maximal p-torus T.
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Theorem (Tₚ)

If T is a p-torus and $H = C^\circ(T)$, then the union of the conjugates of H is generic in G.
**Theorem (Tₚ)**

*If T is a p-torus and \( H = C^\circ(T) \), then the union of the conjugates of H is generic in G.*

**Properties of \( H = C^\circ(T) \):**
- Almost self-normalizing (Rigidity-I)
- Generically disjoint from its conjugates: \( H \setminus (\bigcup H^{G \setminus N(H)}) \) generic in H.
### Theorem (T_p)

If $T$ is a $p$-torus and $H = C^o(T)$, then the union of the conjugates of $H$ is generic in $G$.

### Lemma (Genericity Lemma)

If a definable subgroup $H$ of $G$ is almost self-normalizing and generically disjoint from its conjugates then:

- $\bigcup H^G$ is generic in $G$;
- For $X \subseteq H$, we have $\bigcup X^G$ generic in $G$ if and only if $\bigcup X^H$ is generic in $H$.

### Definition

$X$ is generous in $G$ if the union of its conjugates is generic in $G$. 

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Genericity, Generosity, and Tori
Theorem ($T_p$)
If $T$ is a $p$-torus and $H = C^\circ(T)$, then $H$ is generous in $G$.

Lemma (Genericity Lemma)
If a definable subgroup $H$ of $G$ is almost self-normalizing and generically disjoint from its conjugates then:

- $H$ is generous in $G$;
- For $X \subseteq H$, we have $X$ is generous in $G$ if and only if $X$ is generous in $H$.

Definition
$X$ is generous in $G$ if the union of its conjugates is generic in $G$. 
Problem

Let $G$ be a connected group of finite Morley rank which satisfies the condition

$$x^n = 1$$

generically. Then $G$ satisfies the condition

$$x^n = 1$$

identically.

Theorem

$G$ as above. If $x^n = 1$ generically on $G$, and $n$ is a power of 2, then $x^n = 1$ identically on $G$. 
Problem

Let $G$ be a connected group of finite Morley rank which satisfies the condition

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generically. Then $G$ satisfies the condition

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identically.

More generally:

Theorem

$G$ as above. If $x^n = 1$ generically on $G$, and $n = 2^k n_O$ with $n_O$ odd, then $G = U \ast G_1$ with $U$ a 2-group of bounded exponent and $G/U$ a group satisfying $x^{n_O} = 1$ generically.
Analysis:

- $G$ contains no nontrivial $p$-torus.

- $G = U \ast G_1$ with $U$ a 2-group of bounded exponent and $G/U$ containing no involutions.
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Analysis:

- $G$ contains no nontrivial $p$-torus.
  - $T = d(T_p); H = C^o(T_p)$
  - $x^n = 1$ generically in $G$

- $G = U \star G_1$ with $U$ a 2-group of bounded exponent and $G/U$ containing no involutions.
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Analysis:
- $G$ contains no nontrivial $p$-torus.
  - $T = d(T_p)$; $H = C^\circ(T_p)$
  - $x^n = 1$ generically in $G$
  - $x^n = 1$ generically in $H$
  - $x^n = 1$ generically in $Ta$ some $a \in H$
  - $x^n = 1$ generically in $T$
  - $T = 1$

- $G = U \ast G_1$ with $U$ a 2-group of bounded exponent and $G/U$ containing no involutions.
  - Theorem U
A **Carter subgroup** of $G$ is a connected definable nilpotent subgroup which is almost self-normalizing.

**Theorem (Frécon-Jaligot)**

*They exist.*

**Theorem (Frécon)**

*If the group $G$ involves no bad groups and no bad fields, and $T_0$ is a maximal divisible torsion subgroup, then $C^\circ(T_0)$ is a Carter subgroup.*
Definition

A **Carter subgroup** of $G$ is a connected definable nilpotent subgroup which is almost self-normalizing.

Theorem (Frécon-Jaligot)

*They exist.*

Construction in general:
Let $Q$ be the largest and most semisimple nilpotent subgroup you can find. Then $Q$ is a Carter subgroup.
Carter Subgroups

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One would like to know that the Carter subgroups constructed in this way are generous and are all conjugate. We are taking the 0th approximation to this as our fundamental structural fact.
Carter Subgroups

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One would like to know that the Carter subgroups constructed in this way are generous and are all conjugate. We are taking the $0^{th}$ approximation to this as our fundamental structural fact.

See (or hear) Frécon . . .
Generosity Arguments

Selected Examples

- Degenerate type groups
- Limoncello
- Toricity
Degenerate type groups

Sylow 2-subgroup finite, nontrivial. Minimal example, simple (without loss). Any 2-element will lie *outside* any proper definable connected subgroup of our ambient group $G$. Useful simplification:

**Lemma (EA)**

*The Sylow 2-subgroup of $G$ is elementary abelian.*

Genericity argument

Afterward, other techniques are brought to bear.
Lemma EA

Elementary abelian Sylow 2-subgroup = no elements of order 4.
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\[ t \mapsto H_t \]

- Covariant: \( H_{tg} = H_t^g \)
- Almost selfnormalizing: \( N^o(H_t) = H_t \).
Lemma EA

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- Covariant: \( H_t g = H_t^g \)
- Almost selfnormalizing: \( N^\circ(H_t) = H_t \).

Here \( t \neq 1 \), and \( H_t \) a proper connected definable subgroup for \( t \neq 1 \).

Definition: \( H_t = N^\circ(\ldots N^\circ(C^\circ(t)) \ldots) \). One takes connected normalizers until it stabilizes.

This is only interesting for \( t \) a 2-element, in which case \( t \not\in H_t \) (by minimality).
Lemma EA

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This is only interesting for \( t \) a 2-element, in which case \( t \notin H_t \) (by minimality).

Claim

For any 2-element \( t \neq 1 \), the coset \( tH_t \) is generous.

For \( a \in tH_t \) and \( t \) a 2-element, \( [d(a) : d^\circ(a)] = o(t) \). So the cosets corresponding to \( t \) of order 2 or 4 are disjoint, and our claim follows.
Lemma EA

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- Almost selfnormalizing: \( N^\circ(H_t) = H_t \).

Claim

For any 2-element \( t \neq 1 \), the coset \( tH_t \) is generous.

Proof.

A variation on the standard genericity argument:

- \( N^\circ(tH_t) = H_t \)
- The conjugates of \( tH_t \) are pairwise disjoint
The initial configuration

Even type.
A “uniqueness” case, weak embedding, $M \leq G$ “big”.

\[
M \cap M^g \text{ contains a nontrivial unipotent 2-subgroup iff } g \in M
\]

Aim: $G = SL_2$ (char. 2) and $M$ a Borel subgroup
The initial configuration

Even type.
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Aim: $G = SL_2$ (char. 2) and $M$ a Borel subgroup

$A \leq M$ elementary abelian, $M/C^\circ(A) 2^\perp$.
In fact $A = \Omega_1(O_2^\circ(M))$. 
The initial configuration

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Case division

Subcase 2: \( SL_2 \) sits as a proper subgroup of \( G \).
Technically, we want to shift the line of division to:
Subcase 2*: There are distinct conjugates \( A_1, A_2 \) of \( G \) with
\( H = C^\circ(A_1, A_2) > 1 \).
Then \( L = \langle A_1, A_2 \rangle \leq C^\circ(H) < G \) and this gives us \( L \cong SL_2 < G \).
The initial configuration

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Aim: \( G = SL_2 \) (char. 2) and \( M \) a Borel subgroup

\[
L \cong SL_2 < G.
\]

Case 2*, The main line

\( L \) contains 1-dimensional algebraic tori \( T \)—good tori (Wagner)
We learned in earlier “waves” of analysis that we want to look at
the set \( \mathcal{T} \) of conjugates of \( T \) lying in \( M \), and eventually prove
they are all conjugate under the action of \( M \). This part of the
analysis originally depended on \( M \) being solvable.
**Conjugacy of tori**

\( \mathcal{T} \): some good tori contained in \( M \).

Objective: \( \mathcal{T} \) consists of a single conjugacy class under the action of \( M \).
Conjugacy of tori

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Objective: \( \mathcal{T} \) consists of a single conjugacy class under the action of \( M \).

**Lemma**

Maximal good tori in \( M \) are generous in \( M \), and are conjugate.

**Lemma**

Let \( \mathcal{F} \) be a uniformly definable family of good tori, invariant under conjugation in \( M \). Then \( \mathcal{F} \) breaks up into finitely many \( M \)-conjugacy classes.

**Proof.**

\( T_0 \) a maximal good torus of \( M \).
\( \mathcal{F}_0 \) the set of conjugates of tori in \( \mathcal{F} \) that lie in \( T_0 \).
\( \mathcal{F}_0 \) is a uniformly definable family of subgroups of \( T_0 \), hence finite.
Conjugacy of tori

$\mathcal{T}$: some good tori contained in $M$.
Objective: $\mathcal{T}$ consists of a single conjugacy class under the action of $M$.

**Lemma**

Let $\mathcal{F}$ be a uniformly definable family of good tori, invariant under conjugation in $M$. Then $\mathcal{F}$ breaks up into finitely many $M$-conjugacy classes.

**Proof.**

$T_0$ a maximal good torus of $M$.
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$\mathcal{F}_0$ is a uniformly definable family of subgroups of $T_0$, hence finite.

A little history: The published version of Limoncello runs this way—but the results it quotes are based on arguments found in early drafts of Limoncello.
Groups without unipotent $p$-subgroups

“$p^\perp$-type” (mainly, $p = 2$).

**Theorem**

Let $G$ be a group of finite Morley rank of $p^\perp$ type. Then every $p$-element is $p$-toral (belongs to a $p$-torus).

**Corollary**

Let $G$ be a connected group of finite Morley rank of $p^\perp$ type, and $T$ a maximal $p$-torus. Then every $p$-element $a$ of $C(T)$ belongs to $T$. 
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**Proof.**

$a$ belongs to a maximal torus $T_0$. $T$, $T_0$ are maximal $p$-tori of $C(a)$, hence conjugate in $C(a)$. Forcing $a \in T$. 

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a \in G \ p\text{-element.}
T \text{ a generic maximal } p\text{-torus of } C^\circ(a).
$a \in G$ $p$-element.

$T$ a generic maximal $p$-torus of $C^\circ(a)$.

$$H = C^\circ(a, T)$$

Suppose $a \notin H$. 

$a \in G$ $p$-element.
$T$ a generic maximal $p$-torus of $C^\circ(a)$.

$$H = C^\circ(a, T)$$

Suppose $a \notin H$.

Claim: $Ha$ generous in $G$.

Then generically, $d(g)$ is not $p$-divisible, a contradiction.
a ∈ G p-element.

T a generic maximal p-torus of $C^\circ(a)$.

$$H = C^\circ(a, T)$$

Suppose $a \notin H$.

Claim: $Ha$ generous in $G$.

Then generically, $d(g)$ is not $p$-divisible, a contradiction.

Proof.

Again, $Ha$ turns out to be generically disjoint from its conjugates (in a suitable sense).
(G, X)

Definably primitive: no nontrivial G-invariant definable equivalence relation.

(MPOSA)
(G, X)

**Definably primitive:** no nontrivial G-invariant definable equivalence relation.

(MPOSA)

**Theorem**

(G, X) definably primitive. Then rk (G) is bounded by a function of rk (X).
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Generic transitivity: one large orbit.

Generic \(t\)-transitivity: on \(X^t\).
Theorem

\((G, X)\) definably primitive. Then \(rk\ (G)\) is bounded by a function of \(rk\ (X)\).

Generic transitivity: one large orbit.

Generic \(t\)-transitivity: on \(X^t\).

Proposition

\((G, X)\) definably primitive. Then the degree of multiple transitivity of \(G\) is bounded by a function of \(rk\ (X)\).

(Special case of the theorem, but sufficient.)
Lemma

Let $T$ be an abelian divisible and definable, $T_\infty$ its maximal torsion free definable subgroup of $T$. Then $\text{rk}(T / T_\infty) \leq \text{rk}(X)$.

(In other words, the stabilizer in $T$ of a point of $X$ which is generic over the torsion subgroup is torsion free.)
Lemma

$T$ abelian divisible and definable, $T_\infty$ its maximal torsion free definable subgroup of $T$. Then $\text{rk}(T/T_\infty) \leq \text{rk}(X)$.

Now after reducing to the case of $G$ simple, if $G$ is algebraic this controls the structure of a maximal torus and hence the rank of $G$. 

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Genericity, Generosity, and Tori
Lemma

\[ T \text{ abelian divisible and definable, } T_\infty \text{ its maximal torsion free definable subgroup of } T. \text{ Then } \text{rk} \left( T / T_\infty \right) \leq \text{rk} \left( X \right). \]

Now after reducing to the case of \( G \) simple, if \( G \) is algebraic this controls the structure of a maximal torus and hence the rank of \( G \).

If \( G \) is not algebraic we are in \( 2^\perp \) type and we consider the definable hull \( T \) of a maximal 2-torus (not in \( G \), but in a suitably chosen stabilizer of a small set of points). The generic multiple transitivity gives us an action of \( \text{Sym}_n \).
Lemma

Let $T$ be abelian divisible and definable, $T_\infty$ its maximal torsion free definable subgroup of $T$. Then $\text{rk}(T/T_\infty) \leq \text{rk}(X)$.

Now after reducing to the case of $G$ simple, if $G$ is algebraic this controls the structure of a maximal torus and hence the rank of $G$.

If $G$ is not algebraic we are in $2^\perp$ type and we consider the definable hull $T$ of a maximal 2-torus (not in $G$, but in a suitably chosen stabilizer of a small set of points). The generic multiple transitivity gives us an action of $\text{Sym}_n$.

- If the action is nontrivial then $T/T_\infty$ blows up and we get a contradiction.
- If the action is trivial then we get a 2-element outside $T$ centralizing $T$ and we contradict the corollary to “toricity”.

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Challenges

- Algebraicity of simple $K^*$-groups of odd type
- Absolute bounds on Prüfer rank of groups of odd type
  - Generosity of (some) Carter subgroups
- Construction of bad groups
- Construction of bad field towers.
- Sharp bounds on definably primitive groups
- Explicit classifications of generically 2-transitive actions of simple algebraic groups in the fMr category
- Representation theory of algebraic groups in the fMr category.